Discrete-Time IIR Filter Design from Continuous-Time Filters

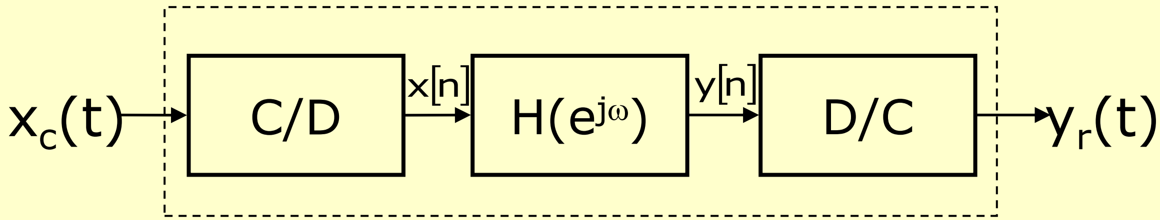
Quote of the Day

Experience is the name everyone gives to their mistakes.

Oscar Wilde

Filter Design Techniques

- Any discrete-time system that modifies certain frequencies
- Frequency-selective filters pass only certain frequencies
- Filter Design Steps
  - Specification
    - Problem or application specific
  - Approximation of specification with a discrete-time system
    - Our focus is to go from spec to discrete-time system
  - Implementation
    - Realization of discrete-time systems depends on target technology
- We already studied the use of discrete-time systems to implement a continuous-time system
  - If our specifications are given in continuous time we can use



$$H\left(e^{j\omega}\right) = H_c\left(j\omega / T\right) \qquad |\omega| < \pi$$

فصل ۷ : روشهای طراحی فیلتر Filter Design

هدف : به دست آوردن $H(z)$ گسسته و علی برای spec مورد نظر

- فیلترهای انتگرال گیر / مشتق گیر : LPF

طراحی فیلتر : فیلتری طراحی کنیم که با داشتن سه ویژگی علی بودن، پایداری و پیاده سازی با ابعاد محدود به فیلتر مورد نظر تا حد ممکن نزدیک باشد. + ضرایب حقیقی

Filter Specifications

مشخصات فیلتر:

- Specifications
  - Passband

باند عبور

$$0.99 \le \left|H_{eff}\left(j\Omega\right)\right| \le 1.01 \quad 0 \le \Omega \le 2\pi\left(2000\right)$$

  - Stopband

باند قطع

$$\left|H_{eff}(j\Omega)\right| \le 0.001 \quad 2\pi(3000) \le \Omega$$

- Parameters

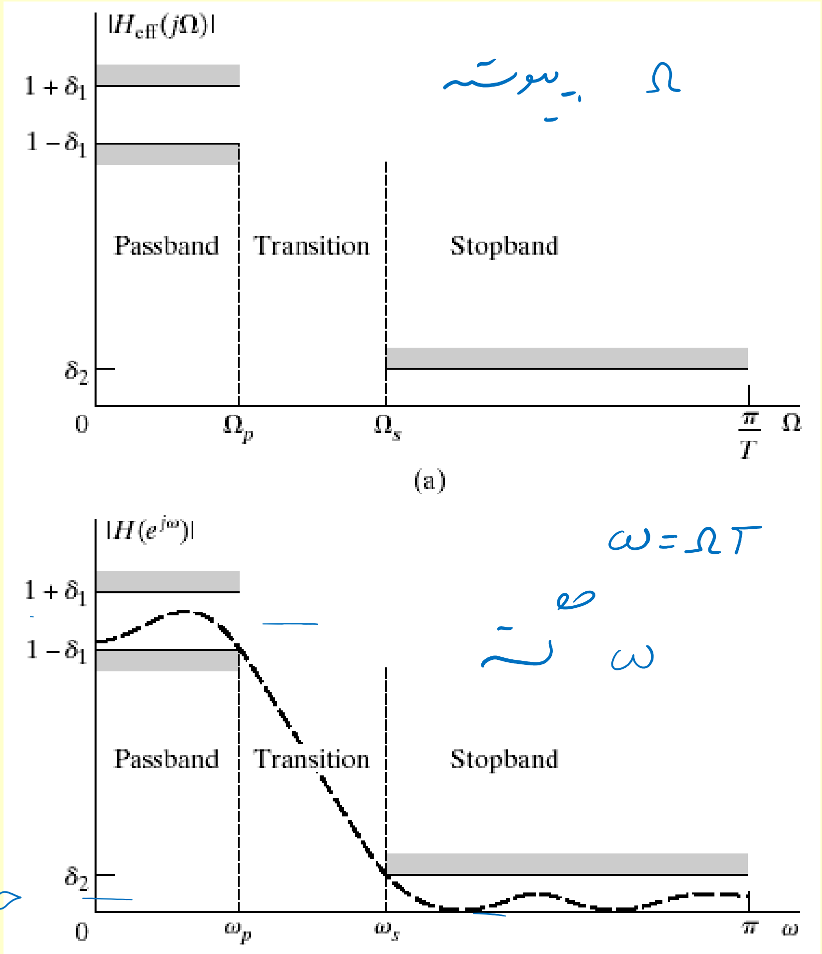
تولرانس باند عبور $\delta_1 = 0.01$ حول یک

باند قطع $\delta_2 = 0.001$

فرکانس قطع عبور $\Omega_p = 2\pi(2000)$

قطع $\Omega_s = 2\pi(3000)$

حول صفر



- Specs in dB
  - Ideal passband gain =20log(1) = 0 dB
  - Max passband gain = 20log(1.01) = 0.086dB
  - Max stopband gain = 20log(0.001) = -60 dB

قضیه وجود: همواره می‌توان $H(z)$ کسری (یا چندجمله‌ای)، علّی، پایدار و حقیقی را یافت که در spec داده شده صدق کند.

اگر δ_1 و δ_2 به صفر نزدیک شوند و $\omega_p \to \omega_s$ میل کند فیلتر به ایده‌آل نزدیک می‌شود ولی در مقابل مرتبه‌ی فیلتر بالا می‌رود.

در مورد فاز: ایده‌آل: FIR با فاز خطی تعمیم یافته

یا طراحی را بر اساس اندازه انجام می‌دهیم، فاز را چک می‌کنیم و دوباره طراحی انجام می‌دهیم.

یا الگوریتم‌هایی را که فاز را در تعریف در نظر می‌گیرند را داشته باشیم که خروجی آنها IIR با فاز خواهد است.

طراحی فیلتر: به جواب یکتا منجر نمی‌شود: زیرا
1- الگوریتم‌های مختلف
2- نوع فیلترهای مختلف
3- مرتبه فیلتر را ما انتخاب کنیم.

هدف: الگوریتم طراحی فیلتر

جزئیات پیاده‌سازی: MATLAB: fdatool (filter design analysis)

7.2 : طراحی زمان گسسته‌ی فیلترهای IIR از فیلترهای پیوسته زمان :

۱، Impulse Invariance

۲، Bilinear Transformation

} IIR

۱، Windowing

۲، Optimum Approximation

} FIR

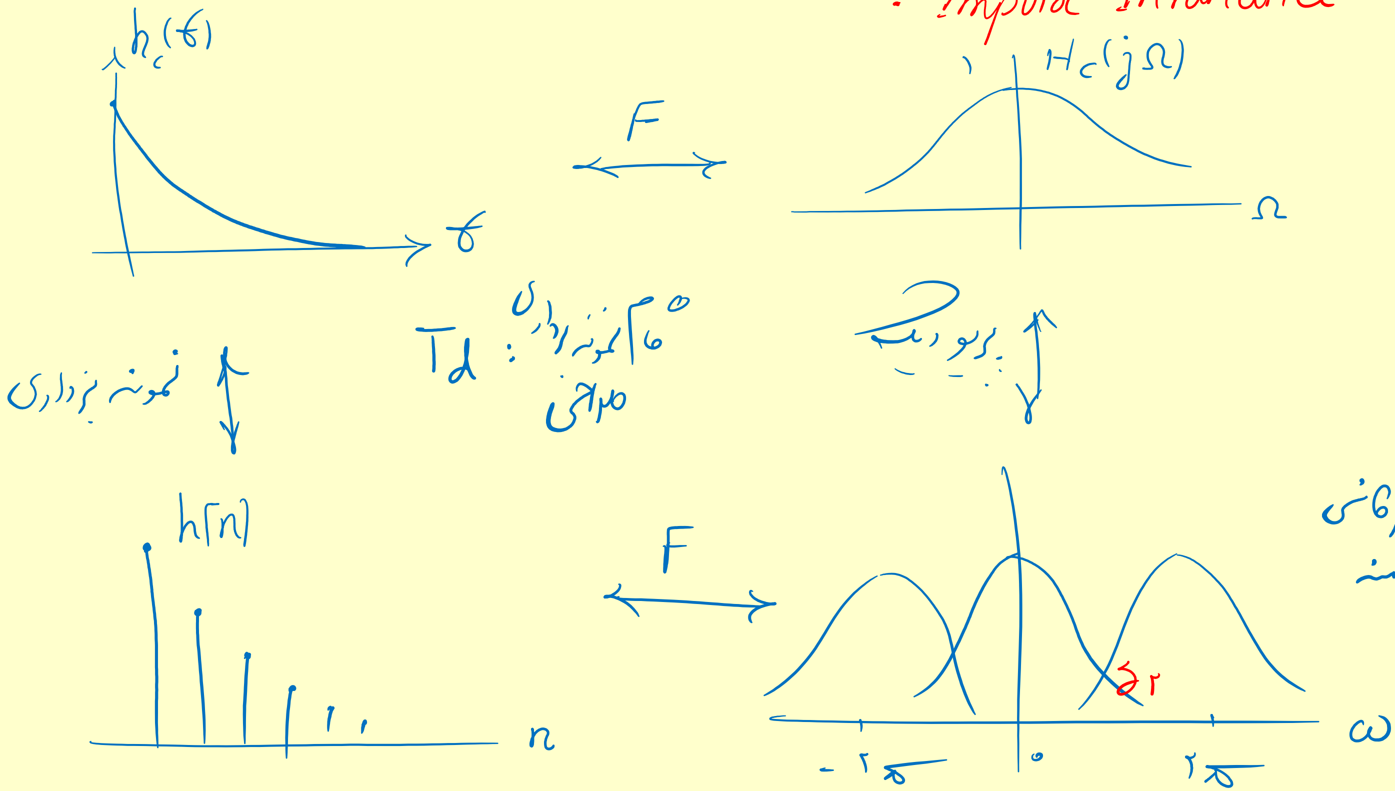
همه‌ی الگوریتم‌ها در MATLAB وجود دارد.

شروع الگوریتم طراحی فیلتر : spec فیلتر : $\delta_1, \delta_2, \omega_p, \omega_s$

توقف " " " : $H(z)$ کسری، علّی، پایدار و حقیقی

چندجمله‌ای

الگوریتم Impulse Invariance :



- تناوب
- Scale فرکانسی
- ضریب دامنه

نقشه: با نمونه برداری از فیلتر پیوسته LP، با فرض محدود بودن aliasing، به فیلتر دیجیتال LP می‌رسیم.

با چه ضریبی شبیه به هم

$$h[n] = T_d \, h_c(nT_d)$$

کافی است فیلتر پیوسته را طراحی کنیم:

الگوریتم

Impulse Invariance

Desired Spec

spec فیلتر گسسته

$\omega_s, \omega_p, \delta_2, \delta_1$

Design spec

spec فیلتر پیوسته

$\Omega_s, \Omega_p, \delta'_2, \delta'_1$

فیلتر ونشتر مدله

طراحی $H_c(s)$

کوزال، پایدار، حقیقی

سخت‌گیرانه‌تر کردن spec طراحی

توجیه کردن که δ_2 نصف

$H(z)$ می‌سازی

کوزال، پایدار، حقیقی

چگونه تبدیل spec گسسته به پیوسته، کوزال، پایدار، حقیقی

$\delta'_1 = \delta_1$

$\delta'_2 = \delta_2$

$\Omega_p = \frac{\omega_p}{T_d}$

$\Omega_s = \frac{\omega_s}{T_d}$

$\boxed{T_d = 1}$

آیا $H(e^{j\omega})$ در spec ها صدق می‌کند؟

به دلیل aliasing

بله

پایان

نحوه‌ی بسط $H(z)$ از روی $H_c(s)$:

$$H_c(s) \rightarrow h(t) \rightarrow h[n] \rightarrow H(z)$$

بررسی کنیم علی، پایدار، حقیقی بودن:

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \xrightarrow{\mathcal{L}^{-1}} h_c(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t)$$

$$h[n] = T_d\, h_c(nT_d) = \sum_{k=1}^{N} A_k T_d\, e^{s_k T_d n} u[n] \rightarrow H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

منفی $Re\{s_k\}$

$|e^{s_k}| < 1 \leftarrow$ داخل دایره‌ی واحد

$\Big\}$ پایدار

علی و حقیقی درست

تبدیل $H_c(s) \sim H(z)$: قطبهای s_k به قطبهای e^{s_k}

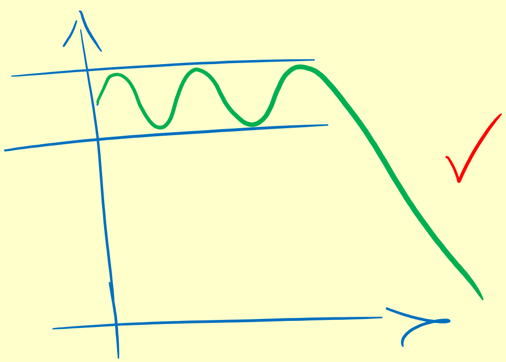
سه عیب الگوریتم Impulse Invariance :

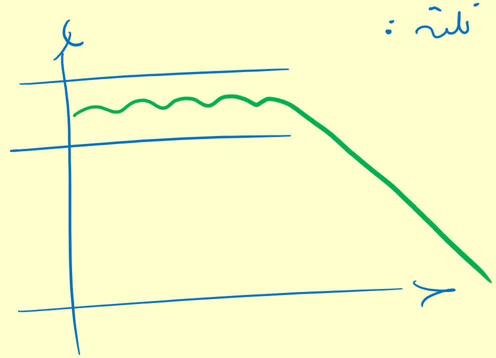
۱- عدم کنترل صفرها

۲_ حلقه در الگوریتم

۳_ ابهام در تحت کنترل ترکیب قطب‌ها که منجر به افزایش مرتبه و هزینه پیاده‌سازی می‌شود.

نکته :





زیرا از همه‌ی درجه‌ها در اختیار استفاده کرده

برتری : شبیه بودن شکل پاسخ ضربه

تا به حال کمتر

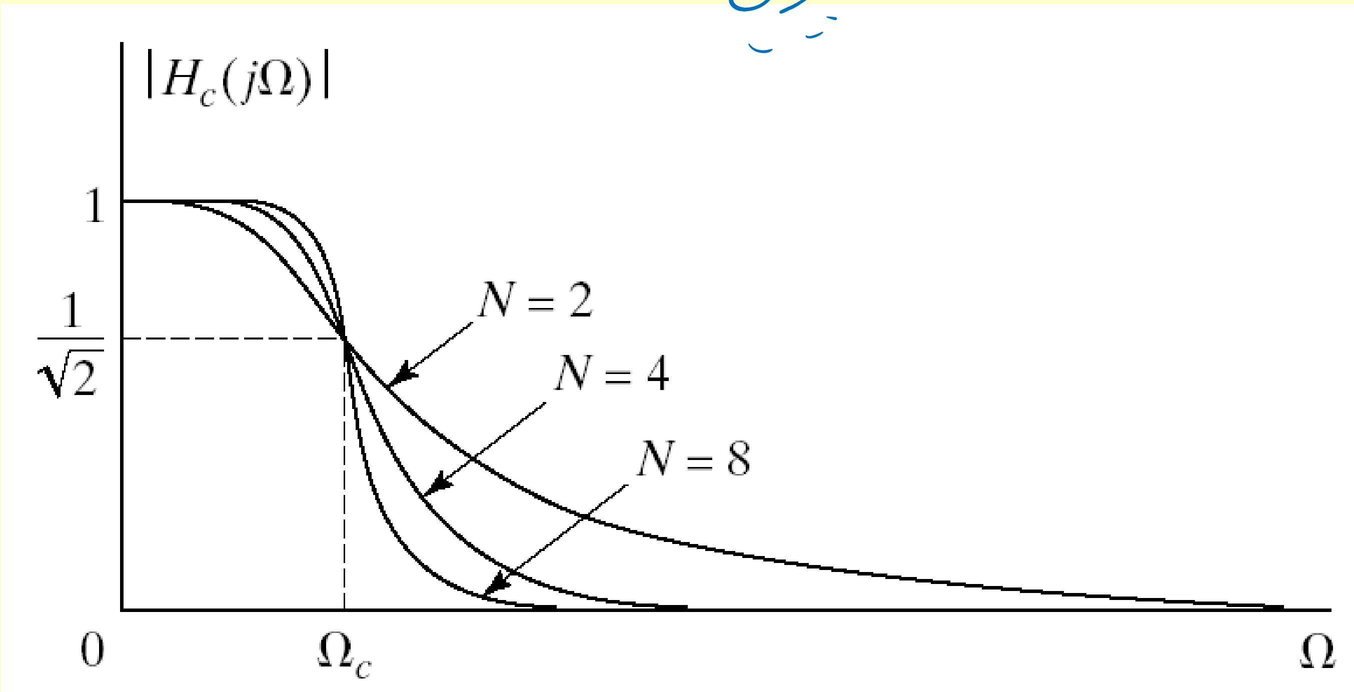
Butterworth Lowpass Filters

- Passband is designed to be maximally flat
- The magnitude-squared function is of the form

$$\boxed{\left|H_c(j\Omega)\right|^2 = \frac{1}{1 + \left(j\Omega / j\Omega_c\right)^{2N}}} \qquad \left|H_c(s)\right|^2 = \frac{1}{1 + \left(s / j\Omega_c\right)^{2N}}$$

N

N, Ω_c



$$s_k = (-1)^{1/2N}(j\Omega_c) = \Omega_c e^{(j\pi/2N)(2k+N-1)} \quad \text{for } k = 0,1,\ldots,2N-1$$

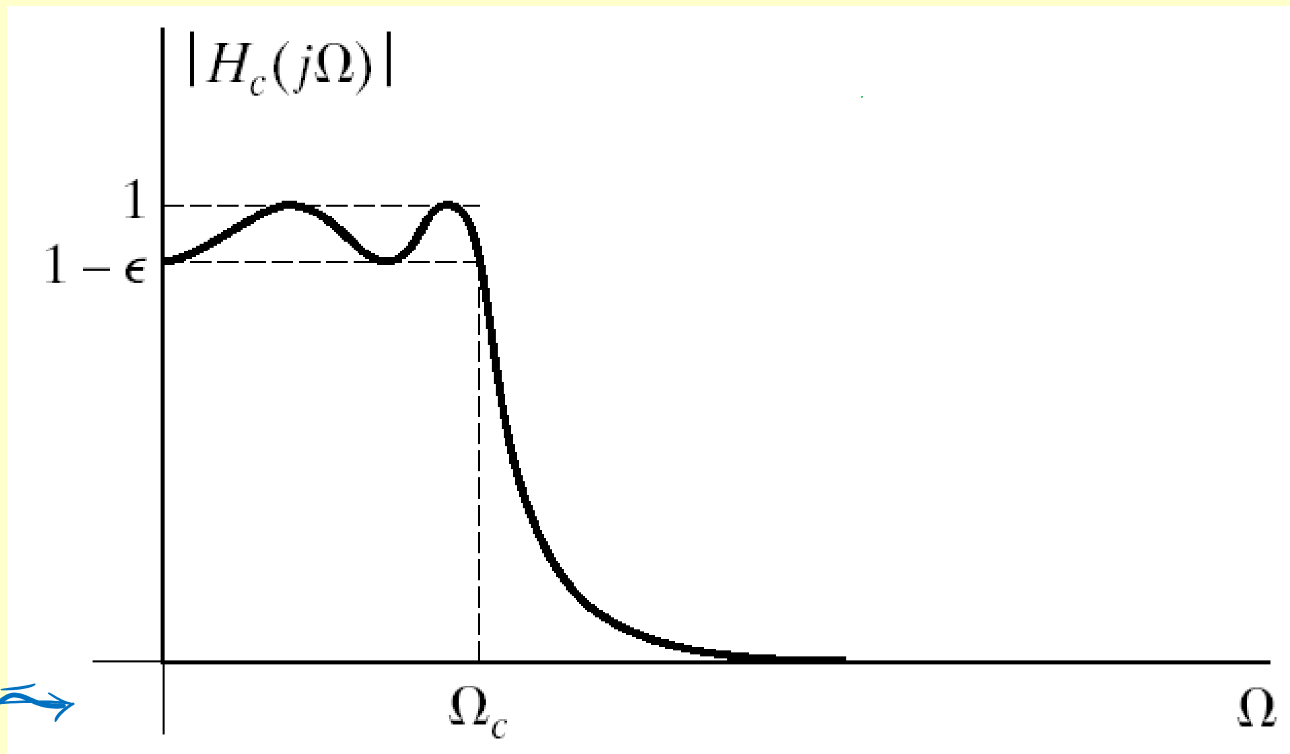
Chebyshev Filters

- Equiripple in the passband and monotonic in the stopband
- Or equiripple in the stopband and monotonic in the passband

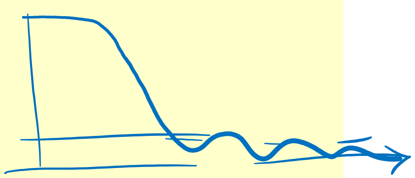
$$|H_c(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 V_N^2(\Omega/\Omega_c)} \qquad V_N(x) = \cos\left(N\cos^{-1}x\right)$$

N, Ω_c, ε

Equiripple



Type II



Elliptic Filters

بهترین N

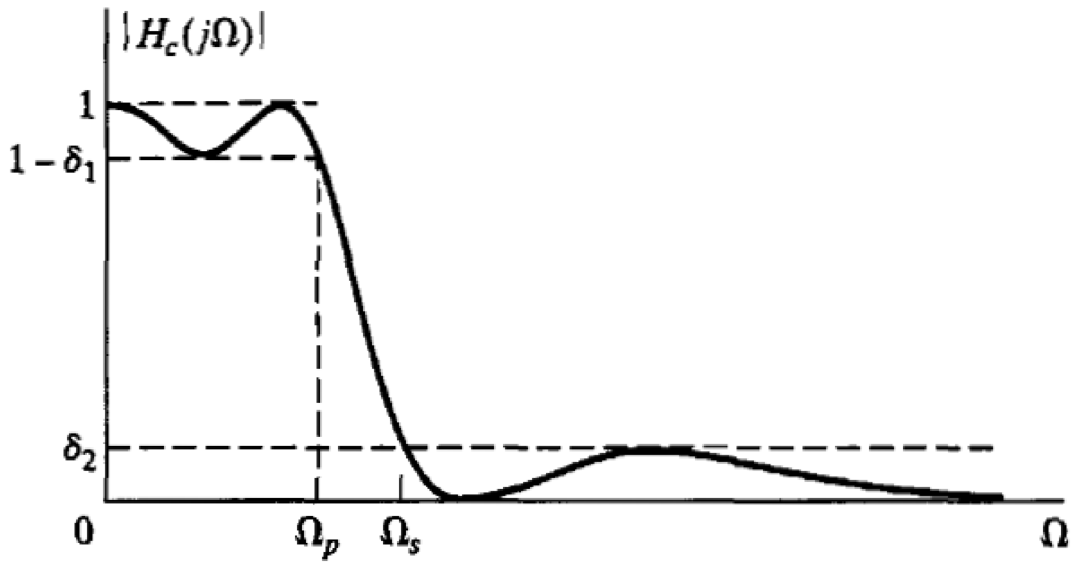


Figure B.6 Equiripple approximation in both passband and stopband.

Filter Design by Impulse Invariance

- Remember impulse invariance
  - Mapping a continuous-time impulse response to discrete-time
  - Mapping a continuous-time frequency response to discrete-time

$$h[n] = T_d h_c(nT_d)$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k\right)$$

- If the continuous-time filter is bandlimited to

$$H_c(j\Omega) = 0 \qquad |\Omega| \geq \pi / T_d$$

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right) \qquad |\omega| \leq \pi$$

- If we start from discrete-time specifications T_d cancels out
  - Start with discrete-time spec in terms of ω
  - Go to continuous-time $\Omega = \omega / T$ and design continuous-time filter
  - Use impulse invariance to map it back to discrete-time $\omega = \Omega T$
- Works best for bandlimited filters due to possible aliasing

Impulse Invariance of System Functions

- Develop impulse invariance relation between system functions
- Partial fraction expansion of transfer function

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

- Corresponding impulse response

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- Impulse response of discrete-time filter

$$h[n] = T_d h_c(nT_d) = \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n] = \sum_{k=1}^{N} T_d A_k \left(e^{s_k T_d}\right)^n u[n]$$

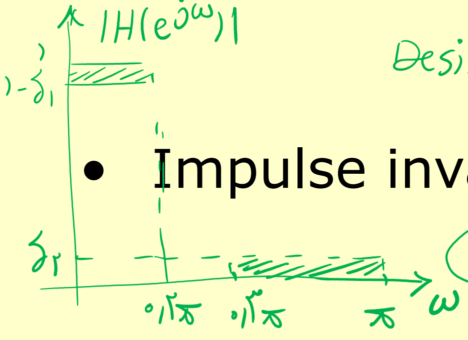
- System function

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

- Pole $s=s_k$ in s-domain transform into pole at $e^{s_k T_d}$

حل:

Desired Spec:



Example

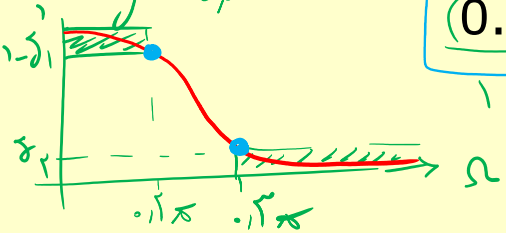
- Impulse invariance applied to Butterworth

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le |\omega| \le 0.2\pi \quad \omega_p$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783 \quad \delta_2 \qquad \omega_s \quad 0.3\pi \le |\omega| \le \pi$$

- Since sampling rate T_d cancels out we can assume $T_d=1$
- Map spec to continuous time

Design Spec:



$$0.89125 \le \left|H(j\Omega)\right| \le 1 \qquad 0 \le |\Omega| \le 0.2\pi \quad \Omega_p$$

$$1-\delta_1$$

$$\left|H(j\Omega)\right| \le 0.17783 \qquad 0.3\pi \le |\Omega| \le \cancel{\pi}$$

$$\delta_2 \qquad \Omega_s$$

- Butterworth filter is monotonic so spec will be satisfied if

داریم:

$$\left|H_c(j0.2\pi)\right| \overset{=}{\ge} 0.89125 \quad \text{and} \quad \left|H_c(j0.3\pi)\right| \overset{=}{\le} 0.17783$$

رابطه فیلتر باترورث با مجهول های Ω_c و N:

$$\left|H_c(j\Omega)\right|^2 = \frac{1}{1+\left(j\Omega / j\Omega_c\right)^{2N}}$$

- Determine N and Ω_c to satisfy these conditions

Example Cont'd

- Satisfy both constrains

$$1+\left(\frac{0.2\pi}{\Omega_c}\right)^{2N}=\left(\frac{1}{0.89125}\right)^2 \quad \text{and} \quad 1+\left(\frac{0.3\pi}{\Omega_c}\right)^{2N}=\left(\frac{1}{0.17783}\right)^2$$

- Solve these equations to get

$$N = 5.8858 \cong 6 \quad \text{and} \quad \Omega_c = 0.70474 \Rightarrow H_c(s)H_c(-s)$$

- N must be an integer so we round it up to meet the spec
- Poles of transfer function

$$s_k = (-1)^{1/12}(j\Omega_c) = \Omega_c e^{(j\pi/12)(2k+11)} \quad \text{for } k = 0,1,\ldots,11$$

علی و پایدار
قطبهای سمت چپ

- The transfer function

$$H_c(s)H_c(-s) = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}$$

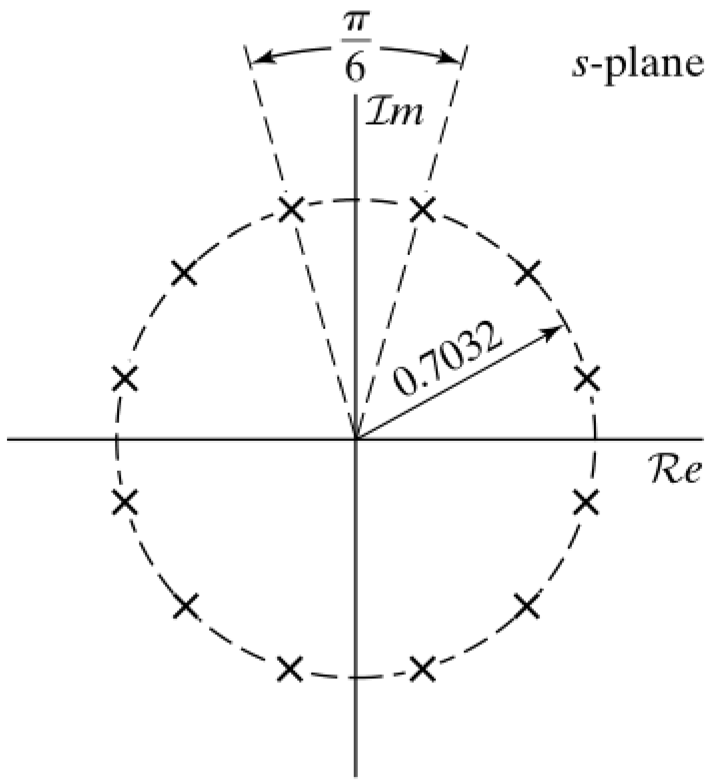
$$H(s)=\frac{0.12093}{\left(s^2+0.364s+0.4945\right)\left(s^2+0.9945s+0.4945\right)\left(s^2+1.3585s+0.4945\right)}$$

- Mapping to z-domain $\quad s_k \longrightarrow e^{s_k}$

$$H(z)=\frac{0.2871-0.4466z^{-1}}{1-1.2971z^{-1}+0.6949z^{-2}}+\frac{-2.1428+1.1455z^{-1}}{1-1.0691z^{-1}+0.3699z^{-2}}$$

$$+\frac{1.8557-0.6303z^{-1}}{1-0.9972z^{-1}+0.257z^{-2}}$$

محل صفر و قطب‌ها



۲π/۱۲ = π/۶

Figure 4 s-plane locations for poles of $H_c(s)H_c(-s)$ for 6^{th}-order Butterworth filter in Example 2.

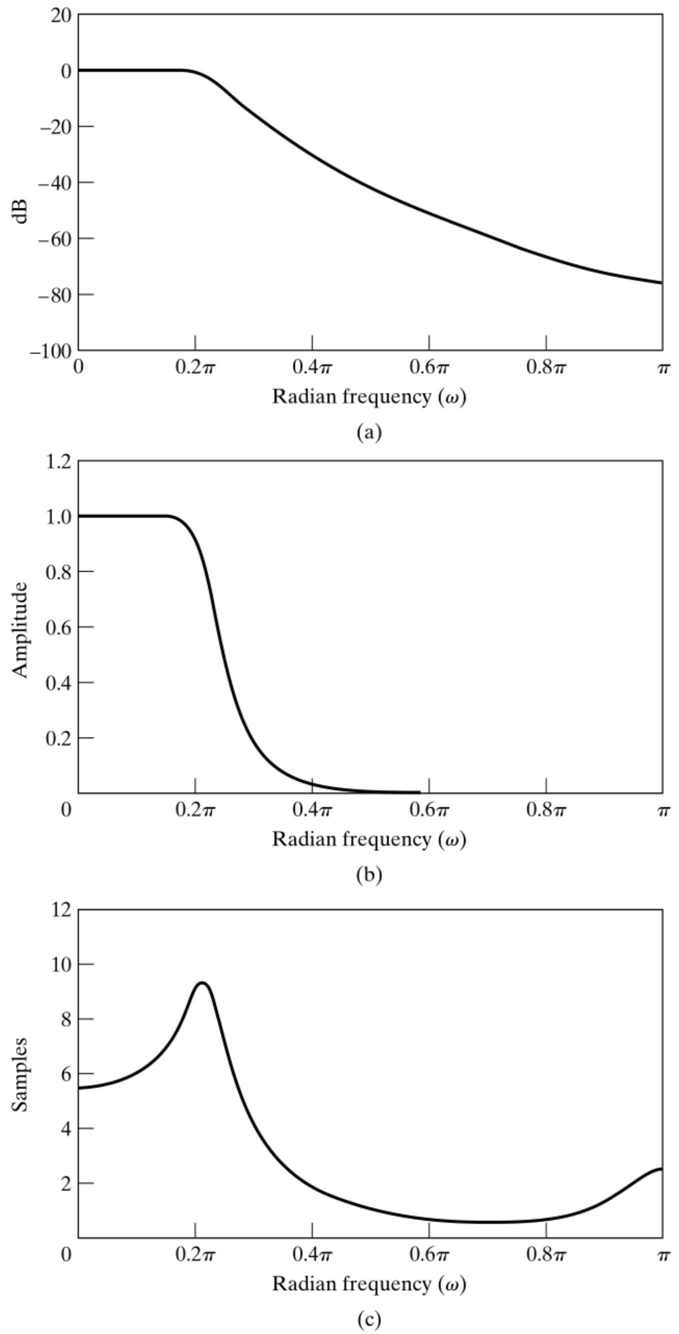
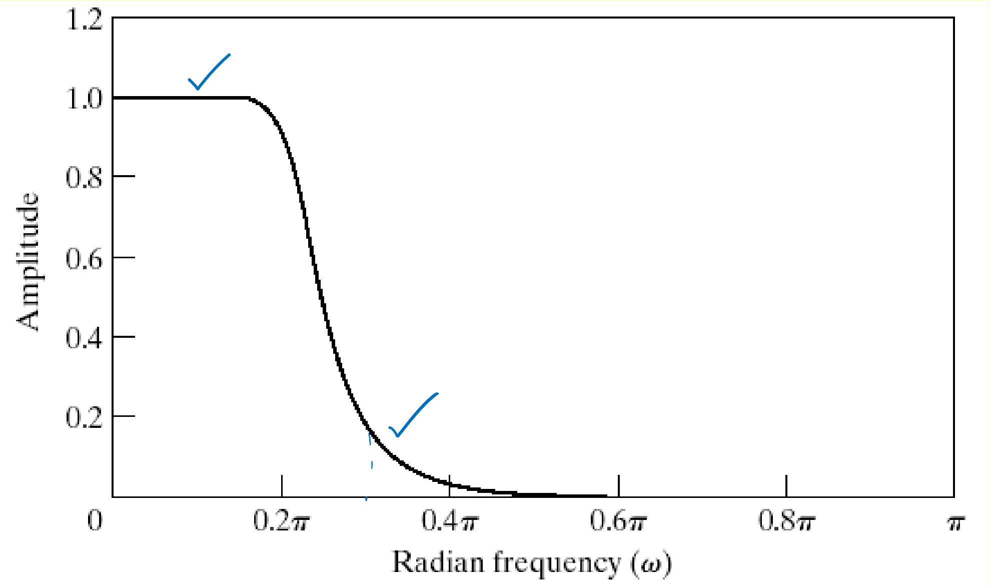
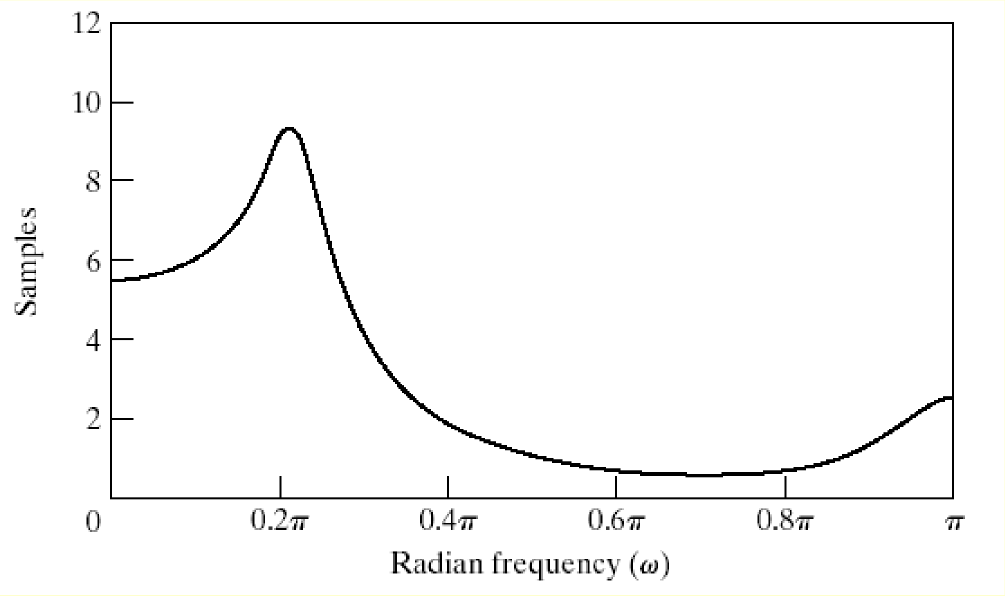


Figure 5 Frequency response of 6th-order Butterworth filter transformed by impulse invariance. (a) Log magnitude in dB. (b) Magnitude. (c) Group delay.

Example Cont'd





Bilinear Transformation الگوریتم

الگوریتم دوم طراحی IIR : تبدیل دوسویه

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

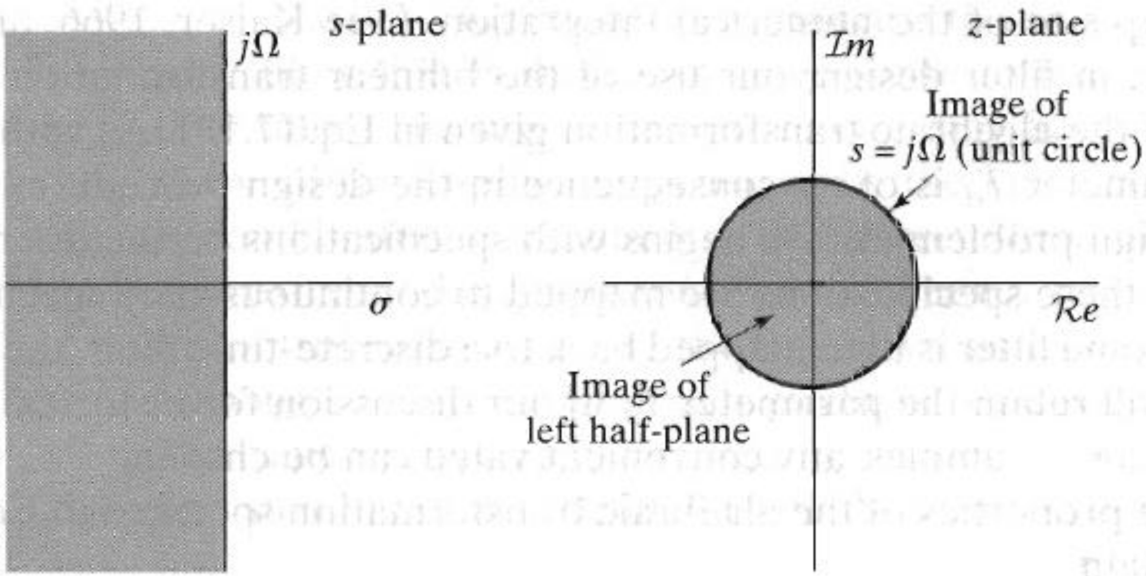


Figure 7.6 Mapping of the *s*-plane onto the *z*-plane using the bilinear transformation.

Filter Design by Bilinear Transformation

- Get around the aliasing problem of impulse invariance
- Map the entire s-plane onto the unit-circle in the z-plane
  - Nonlinear transformation
  - Frequency response subject to warping

اگر $H_c(s)$ کسری، علی، پایدار و حقیقی باشد.
$H(z)$ نیز کسری، علی، پایدار و حقیقی خواهد بود.

- Bilinear transformation

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

- $H_c(s)$ کسری، تبدیل کسری $\Rightarrow$ $H(z)$ کسری
- $H_c(s)$ علی، پایدار $\rightarrow$ قطب‌ها سمت چپ محور $j\omega$
$\rightarrow$ $H(z)$ علی و پایدار $\Leftarrow$ قطب‌های $H(z)$ داخل دایره واحد

- Transformed system function

$$H(z) = H_c(s)\Big|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \Rightarrow H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$$

- $H_c(s)$ حقیقی و تبدیل با ضرایب حقیقی
$\Downarrow$
$H(z)$ حقیقی

- Again T_d cancels out so we can ignore it
- We can solve the transformation for z as

$$z = \frac{1+(T_d/2)s}{1-(T_d/2)s} = \frac{1+\sigma T_d/2 + j\Omega T_d/2}{1-\sigma T_d/2 - j\Omega T_d/2} \qquad s = \sigma + j\Omega$$

- Maps the left-half s-plane into the inside of the unit-circle in z
  - Stable in one domain would stay in the other

Desired spec:

$\delta_1, \delta_2, \omega_p, \omega_s$ زمان گسسته

روابط پایین

$\downarrow$

Design spec:

$\delta_1', \delta_2', \Omega_p, \Omega_s$ زمان پیوسته

$\downarrow$

$H_c(s)$ طراحی

$\downarrow$

$H(z)$ می‌شه

$\downarrow$

پایان

معایب:

۱- تبدیل غیرخطی

۲- پاسخ ضربه حوزه زمان الزاما شبیه به هم نیست.

الگوریتم تبدیل دوسویه:

مزیت ها:

۱- حلقه در الگوریتم ندارد.

۲- ابهام در الگوریتم ندارد.

۳- هم روی صفرها و هم روی قطبها کنترل داریم.

۴- aliasing نداریم.

۵- $H(z)$ به دست می‌آید.

۶- بالاگذر بودن HPF مراعات کرده.

در روش قبلی، HPF بودن را مراعات نکرد زیرا aliasing بسیار زیاد می‌شود.

$$H(z) = H_c(s)\Big|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$s = j\Omega, \quad z = e^{j\omega} \;\longrightarrow\; j\Omega = \frac{2}{T_d}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right) \Rightarrow \begin{cases} \omega = 2\arctan\left(\frac{\Omega T_d}{2}\right) \\ \Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right) \end{cases}$$

تبدیل spec ها:

$$\Rightarrow \begin{cases} \Omega_p = \frac{2}{T_d}\tan\left(\frac{\omega_p}{2}\right) \\ \Omega_s = \frac{2}{T_d}\tan\left(\frac{\omega_s}{2}\right) \end{cases}$$

Bilinear Transformation

- On the unit circle the transform becomes

$$z = \frac{1 + j\Omega T_d / 2}{1 - j\Omega T_d / 2} = e^{j\omega}$$

- To derive the relation between ω and Ω

$$s = \frac{2}{T_d}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right) = \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2} j\sin(\omega/2)}{2e^{-j\omega/2}\cos(\omega/2)}\right] = \frac{2j}{T_d}\tan\left(\frac{\omega}{2}\right)$$

- Which yields

$$\Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right) \quad \text{or} \quad \omega = 2\arctan\left(\frac{\Omega T_d}{2}\right)$$

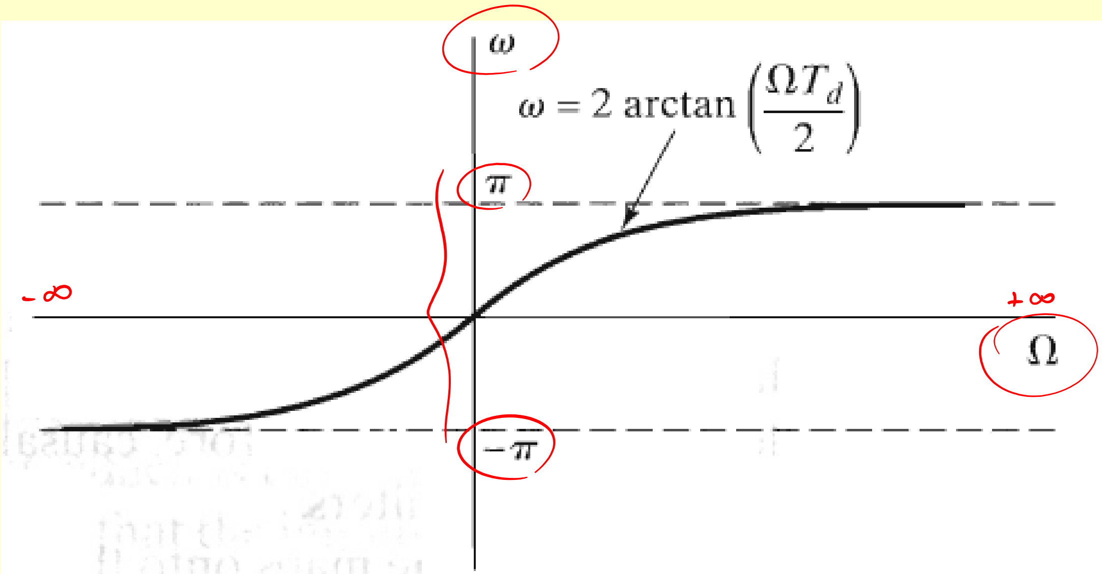


Figure 7.7 Mapping of the continuous-time frequency axis onto the discrete-time frequency axis by bilinear transformation.

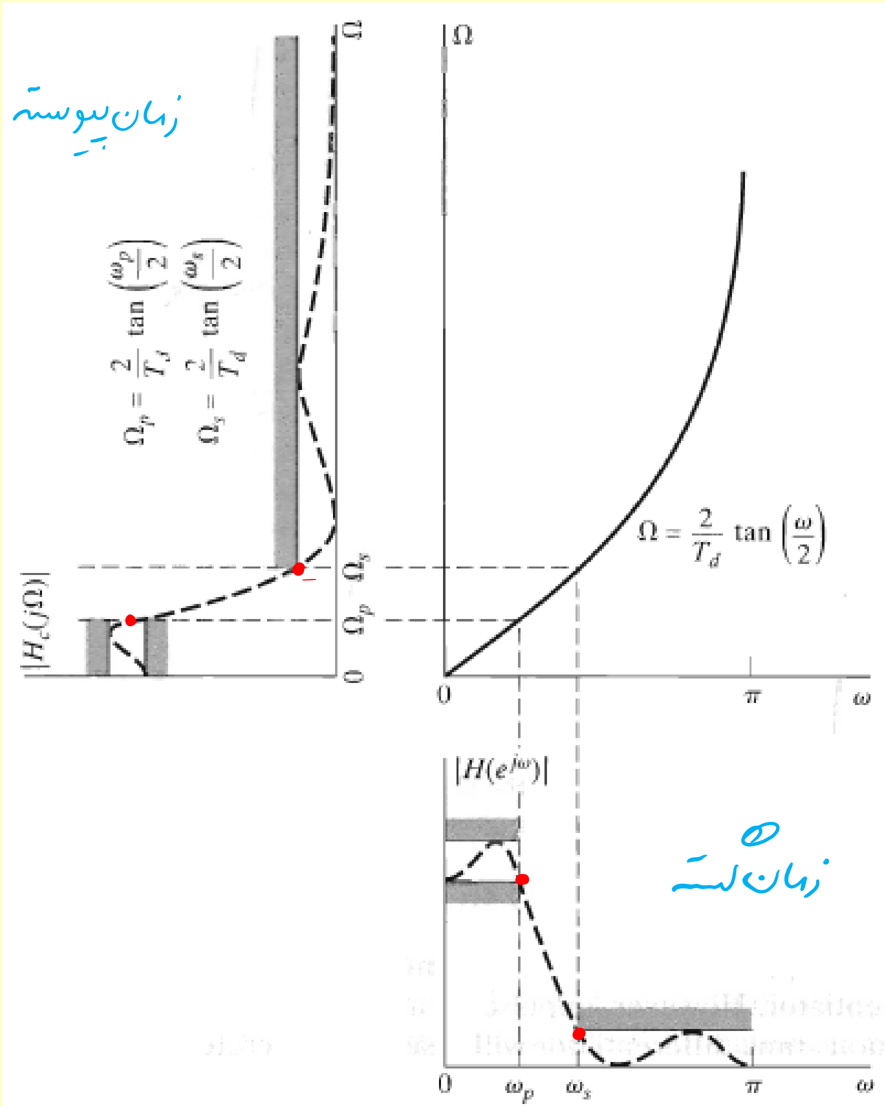
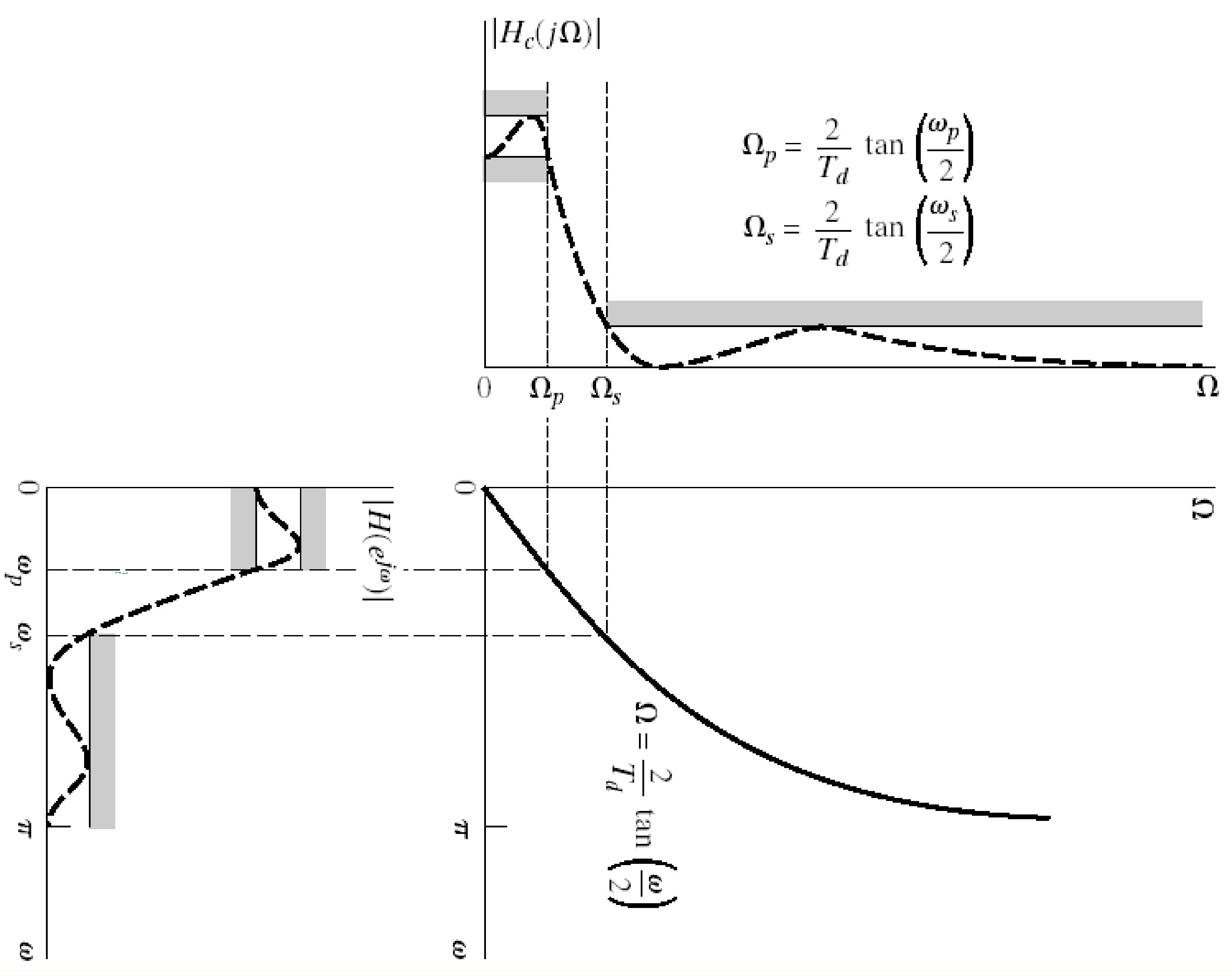


Figure 7.8 Frequency warping inherent in the bilinear transformation of a continuous-time lowpass filter into a discrete-time lowpass filter. To achieve the desired discrete-time cutoff frequencies, the continuous-time cutoff frequencies must be prewarped as indicated.

Bilinear Transformation



Example

حل مثال قبل با روش تبدیل دوسویه:

- Bilinear transform applied to Butterworth

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le |\omega| \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783 \qquad 0.3\pi \le |\omega| \le \pi$$

Desired spec

- Apply bilinear transformation to specifications

$$0.89125 \le \left|H(j\Omega)\right| \le 1 \qquad 0 \le |\Omega| \le \underbrace{\frac{2}{T_d}\tan\left(\frac{0.2\pi}{2}\right)}_{\Omega_p}$$

$$\left|H(j\Omega)\right| \le 0.17783 \qquad \underbrace{\frac{2}{T_d}\tan\left(\frac{0.3\pi}{2}\right)}_{\Omega_s} \le |\Omega| < \infty$$

Design spec

- We can assume T_d=1 and apply the specifications to

باترورث

$$\left|H_c(j\Omega)\right|^2 = \frac{1}{1+\left(\Omega/\Omega_c\right)^{2N}}$$

- To get

$$1+\left(\frac{2\tan 0.1\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \quad \text{and} \quad 1+\left(\frac{2\tan 0.15\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

Example Cont'd

- Solve N and Ω_c

$$N = \frac{\log\left[\left(\left(\frac{1}{0.17783}\right)^2 - 1\right)\Big/\left(\left(\frac{1}{0.89125}\right)^2 - 1\right)\right]}{2\log[\tan(0.15\pi)/\tan(0.1\pi)]} = 5.305 \cong 6 \rightarrow \quad \Omega_c = 0.766$$

N عدد صحيح

- The resulting transfer function has the following poles

$$s_k = (-1)^{1/12}(j\Omega_c) = \Omega_c e^{(j\pi/12)(2k+11)} \quad \text{for } k = 0,1,\ldots,11$$

- Resulting in

$$H_c(s) = \frac{0.20238}{(s^2 + 0.3996s + 0.5871)(s^2 + 1.0836s + 0.5871)(s^2 + 1.4802s + 0.5871)}$$

- Applying the bilinear transform yields

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \quad T_d = 1$$

$$H(z) = \frac{0.0007378(1 + z^{-1})^6}{(1 - 1.2686z^{-1} + 0.7051z^{-2})(1 - 1.0106z^{-1} + 0.3583z^{-2})}$$

$$\times \frac{1}{(1 - 0.9044z^{-1} + 0.2155z^{-2})}$$

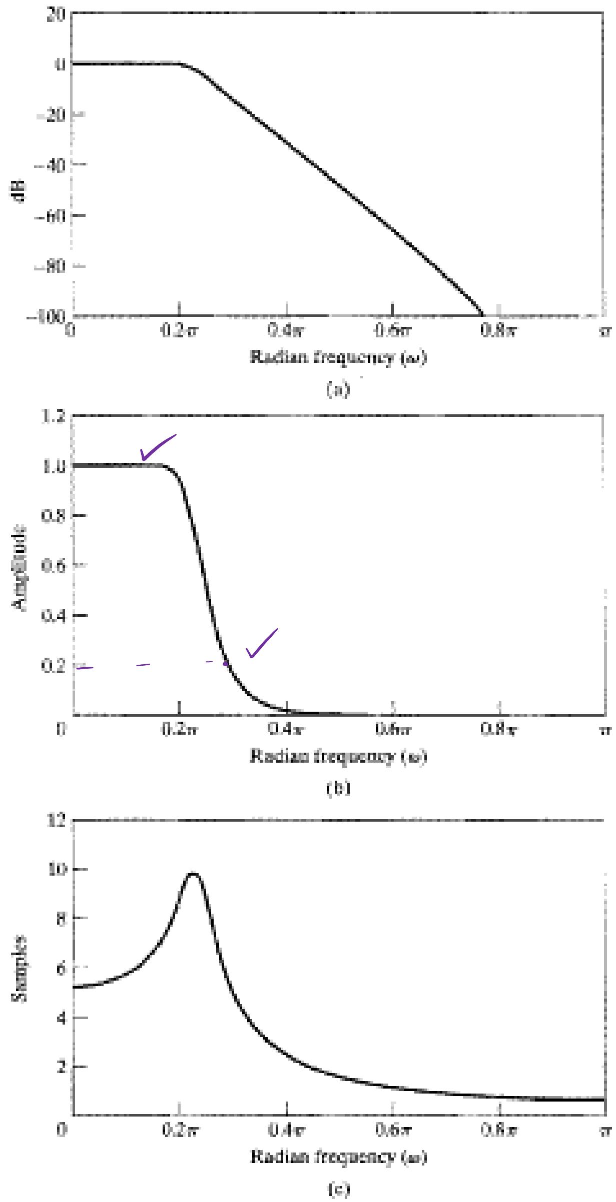
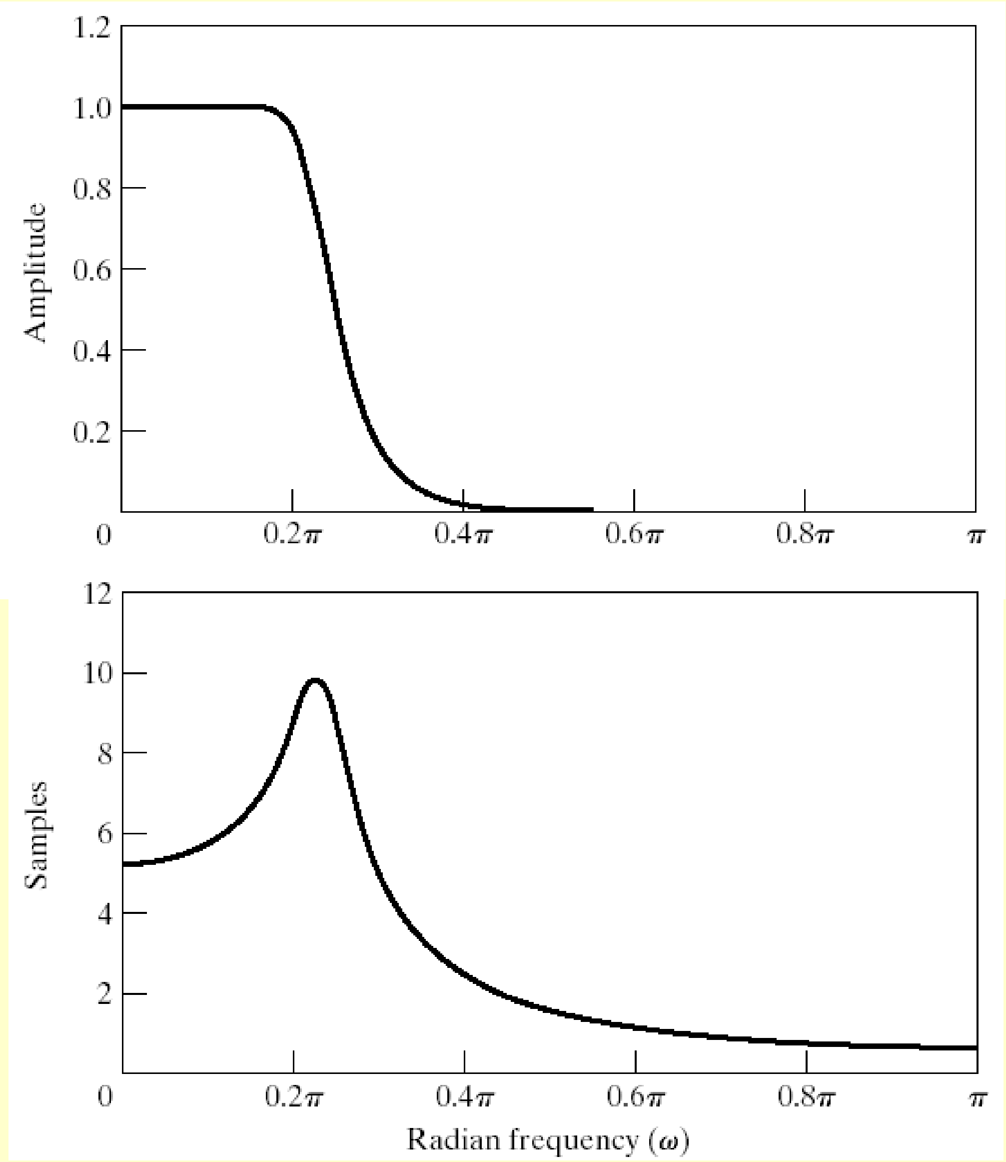


Figure 7.11 Frequency response of 6th-order Butterworth filter transformed by bilinear transform. (a) Log magnitude in dB. (b) Magnitude. (c) Group delay.

Example Cont'd



* مثال 7.4 نیز مطالعه شود.

مثال ۷.۵

$$\begin{cases} \omega_p = 0.4\pi \\ \omega_s = 0.6\pi \\ \delta_1 = 0.01 \\ \delta_2 = 0.001 \end{cases}$$

$0.99 \leq |H(e^{j\omega})| \leq 1.01 \; ; \; |\omega| \leq 0.4\pi$

$|H(e^{j\omega})| \leq 0.001 \; ; \; 0.6\pi \leq |\omega| \leq \pi$

باترورث

مرتبه ۱۴ ← هزینه زیاد

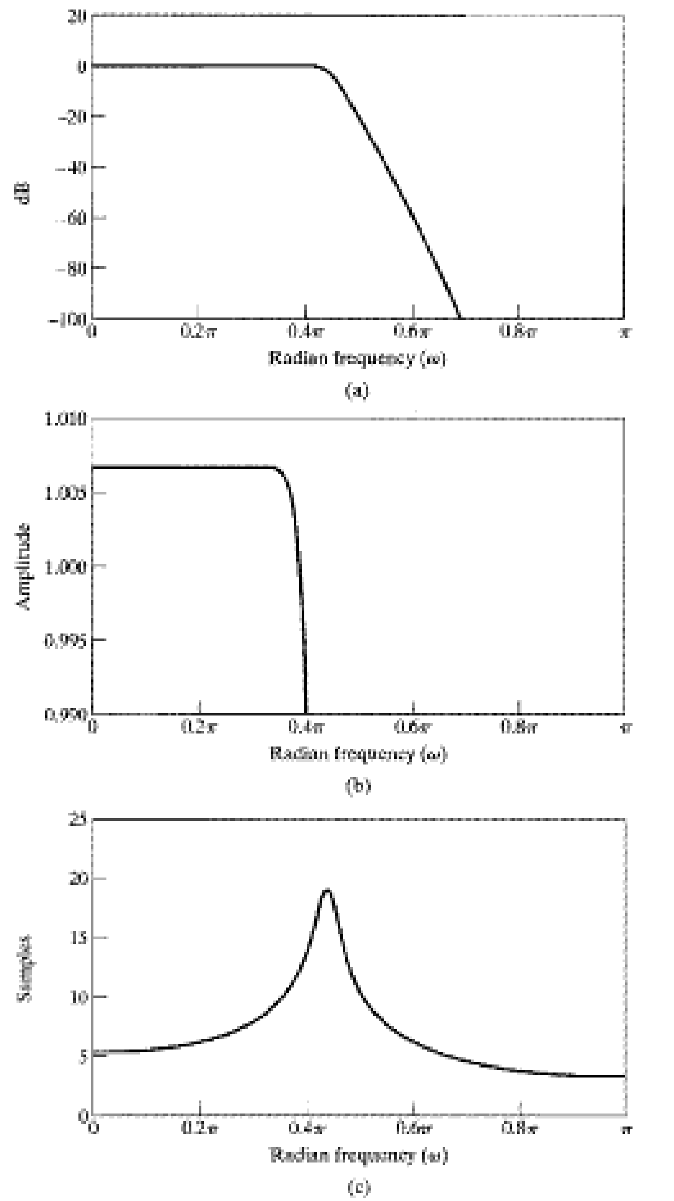


Figure 7.17 Frequency response of 14th-order Butterworth filter in Example 7.5. (a) Log magnitude in dB. (b) Detailed plot of magnitude in passband. (c) Group delay.

محل صفر و قطبها ← فیلتر باترورث مرتبه ۱۴

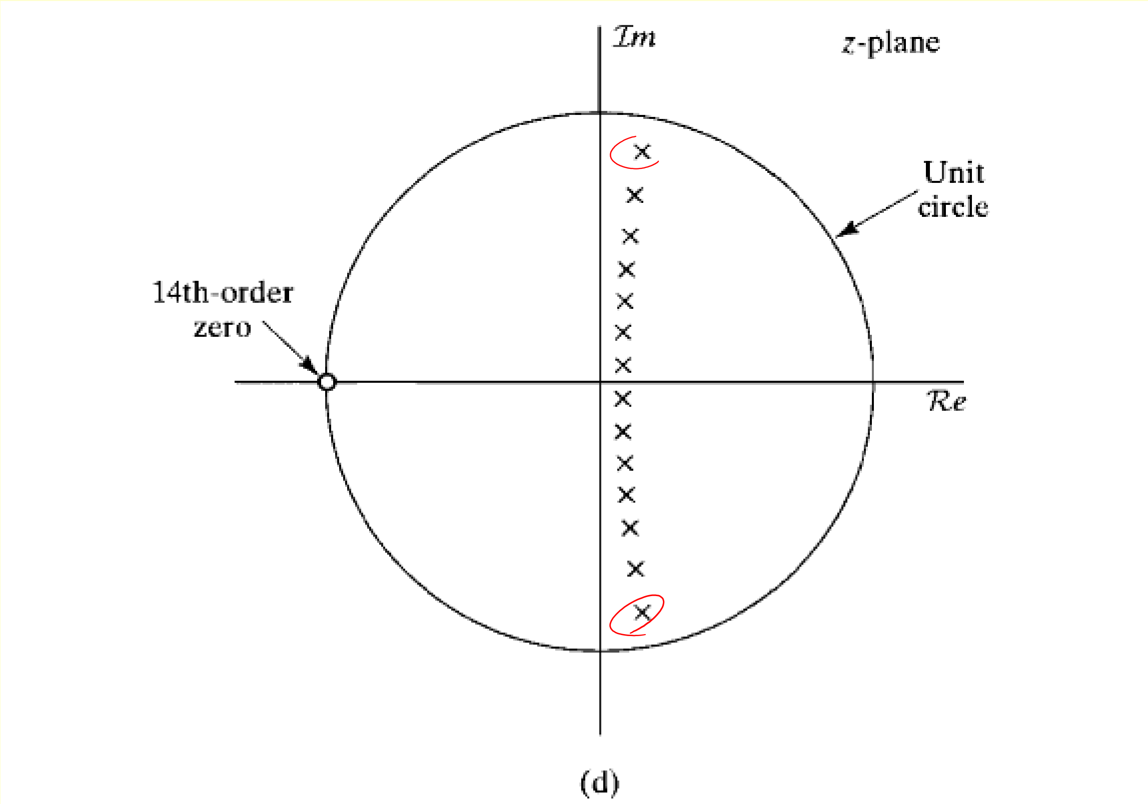


Figure 7.17 (*continued*) (d) Pole–zero plot of 14^{th}-order Butterworth filter in Example 7.5.

چبیشف نوع (۱)

مرتبه ۸

فاز نامطلوب

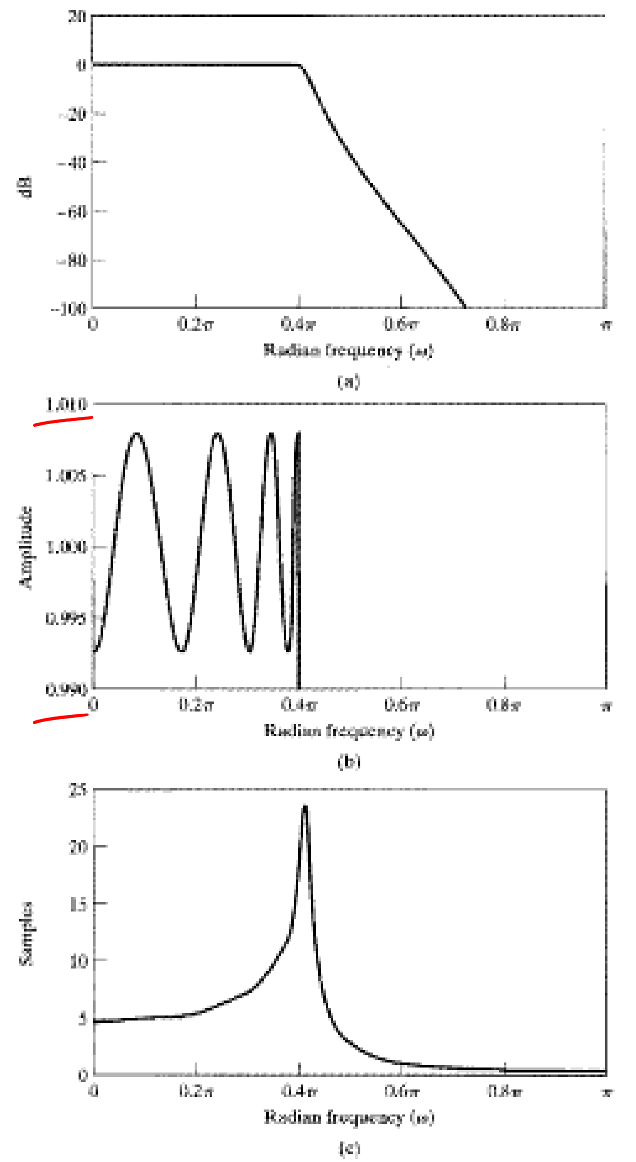


Figure 7.18 Frequency response of 6[th]-order Chebyshev type I filter in Example 7.5. (a) Log magnitude in dB. (b) Detailed plot of magnitude in passband. (c) Group delay.

چبیشف نوع (II)
مرتبهٔ 8

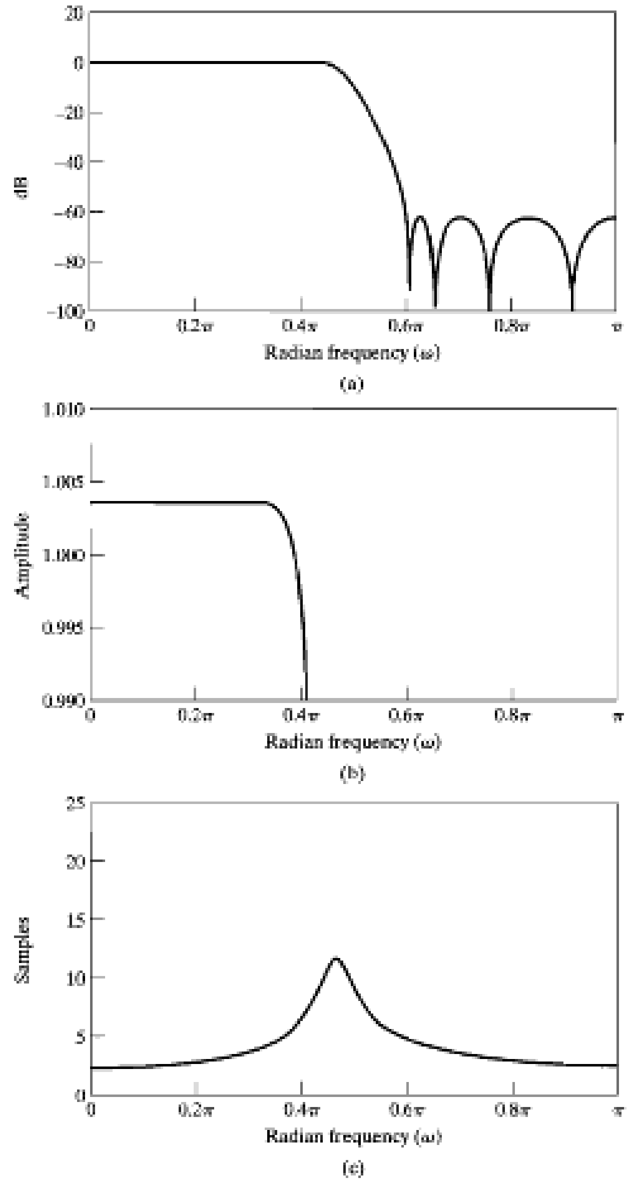


Figure 7.19 Frequency response of 8th-order Chebyshev type II filter in Example 7.5. (a) Log magnitude in dB. (b) Detailed plot of magnitude in passband. (c) Group delay.

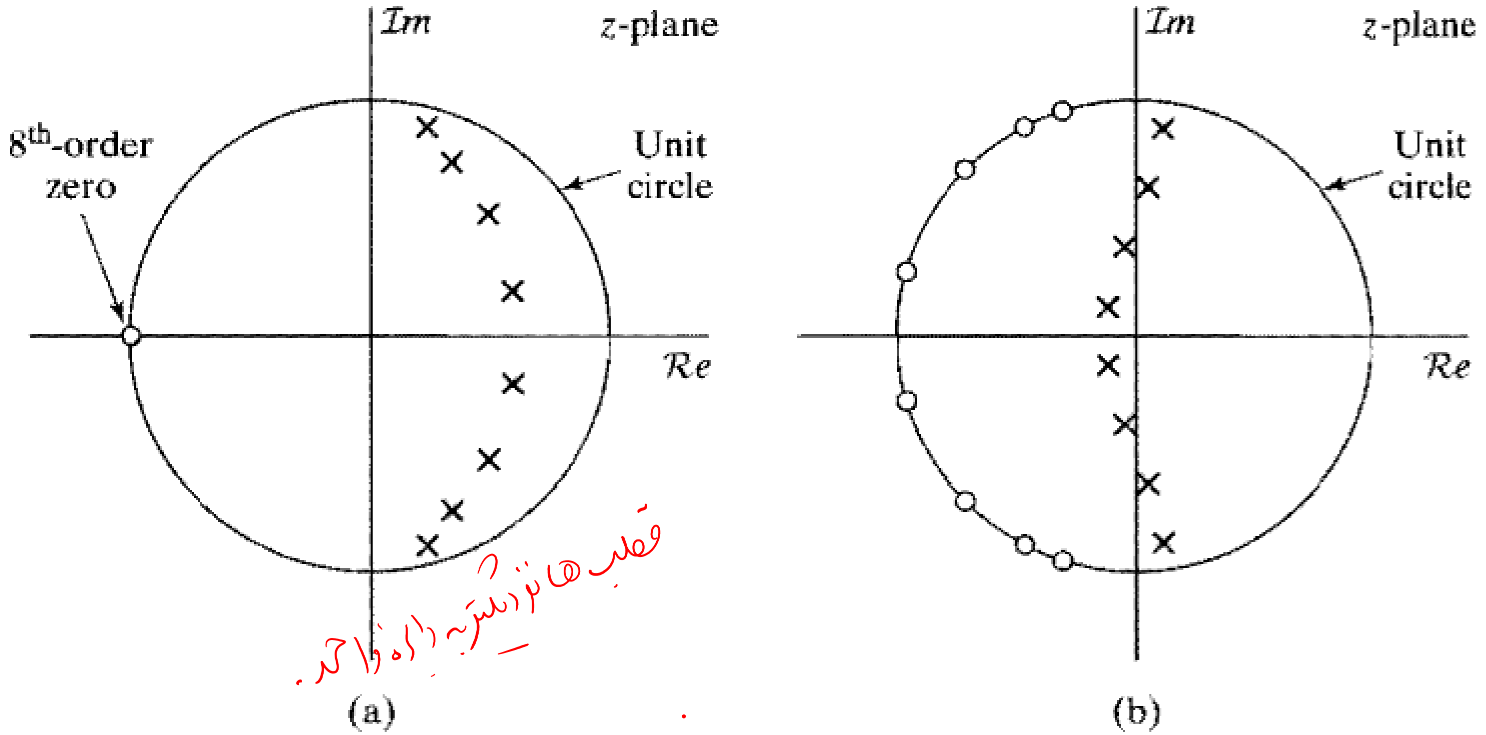


Figure 7.20 Pole–zero plot of 8^{th}-order Chebyshev filters in Example 7.5. (a) Type I. (b) Type II.

Elliptic

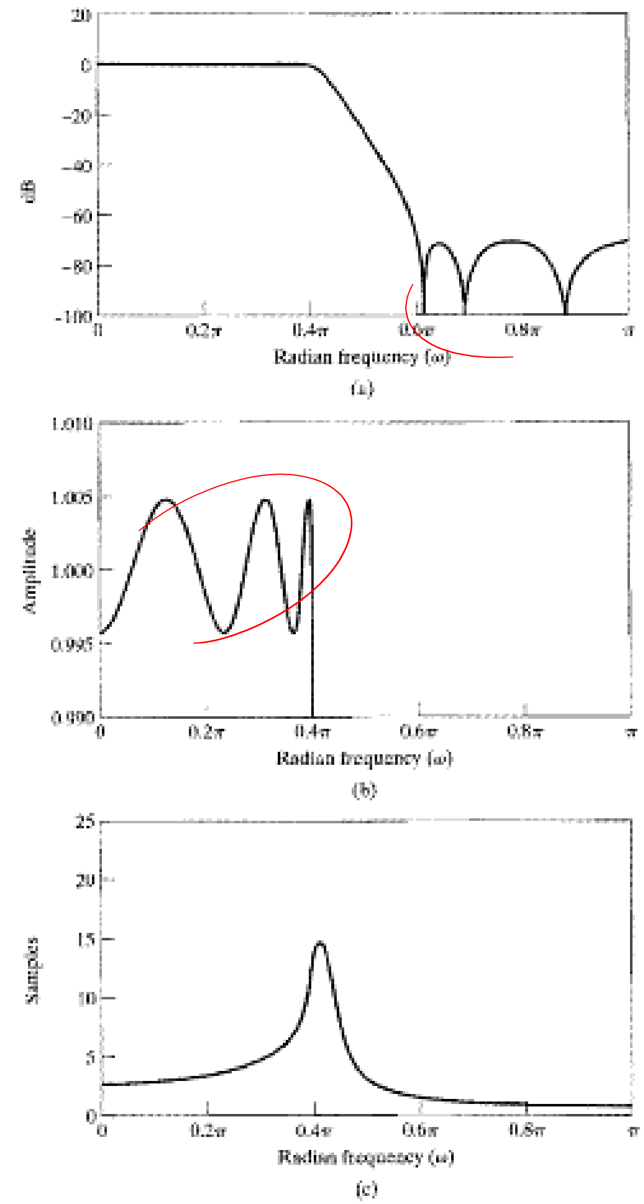


Figure 7.21 Frequency response of 6[th]-order elliptic filter in Example 7.5. (a) Log magnitude in dB. (b) Detailed plot of magnitude in passband. (c) Group delay.

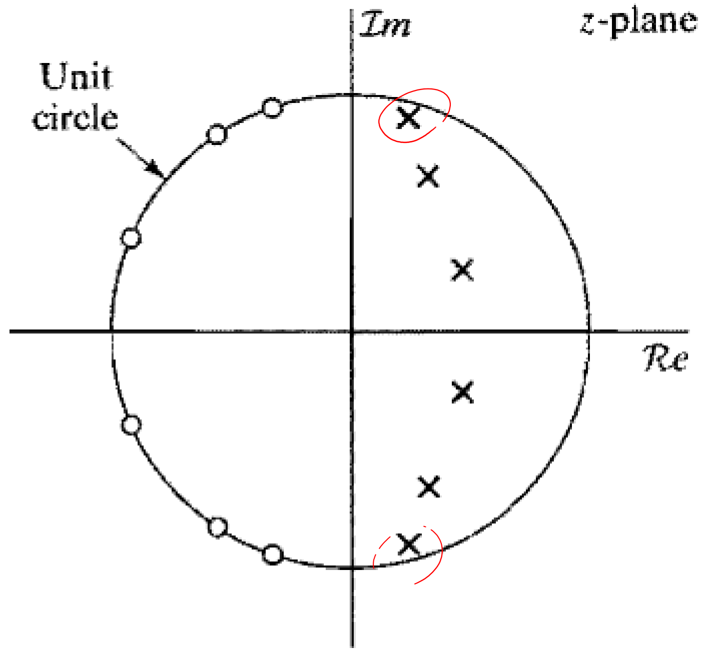


Figure 7.22 Pole–zero plot of 6th-order elliptic filter in Example 7.5.

Multiband

تبدیل LPF به چندباندە :

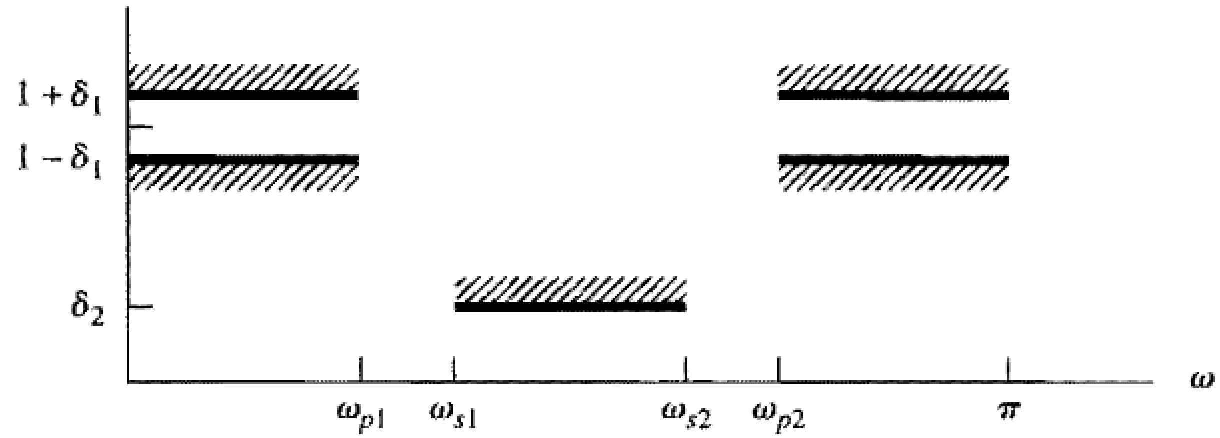


Figure 7.23 Tolerance scheme for a multiband filter.

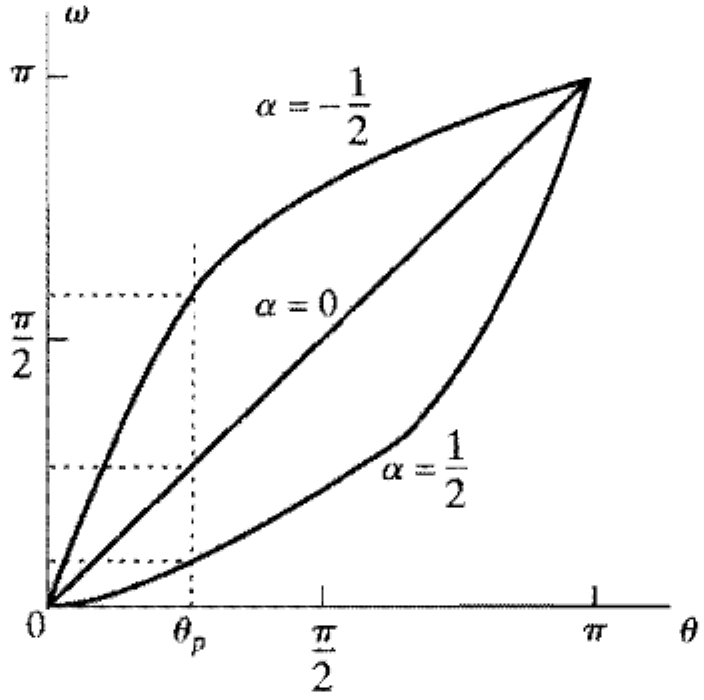


Figure 7.24 Warping of the frequency scale in lowpass-to-lowpass transformation.

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \dfrac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
✓ Highpass	$Z^{-1} = -\dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\dfrac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ ω_p = desired cutoff frequency ← فرکانس قطع فیلتر بالاگذر
✓ Bandpass	$Z^{-1} = -\dfrac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\dfrac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency
✓ Bandstop	$Z^{-1} = \dfrac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\dfrac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency

$H(z) = H_{LP}(Z)\big|$

HPF

$Z^{-1} = G(z^{-1})$

پایین گذر

7.6

پایین گذر به بالا گذر

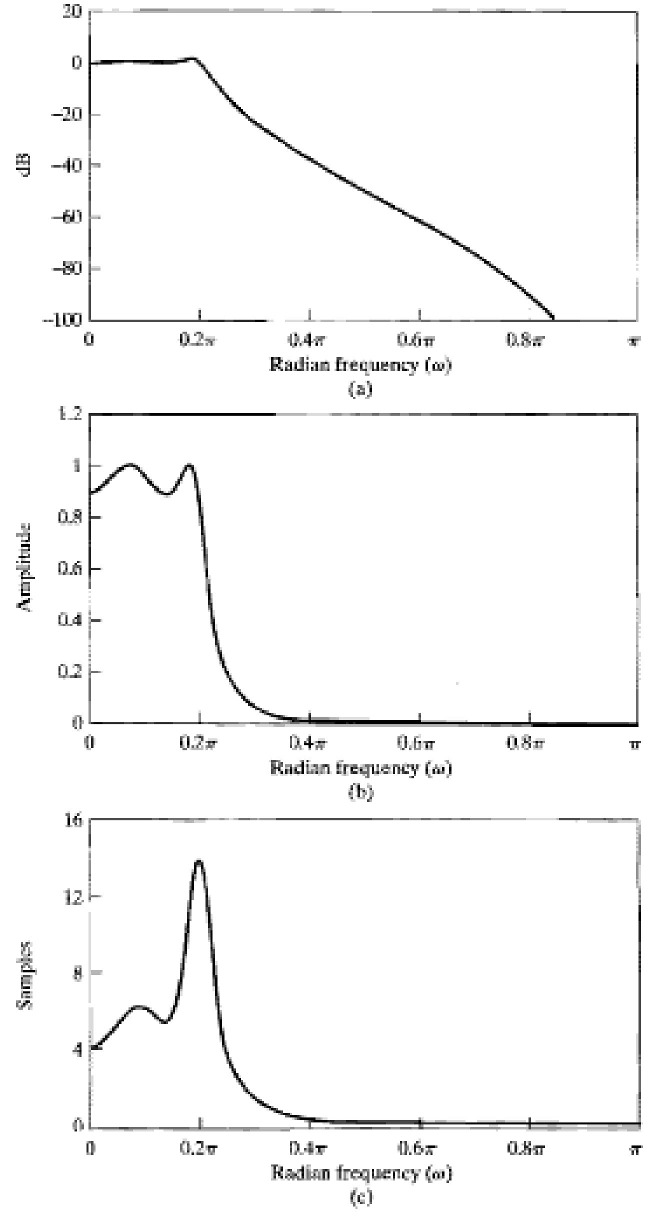


Figure 7.25 Frequency response of 4th-order Chebyshev lowpass filter. (a) Log magnitude in dB. (b) Magnitude. (c) Group delay.

بالاگذر

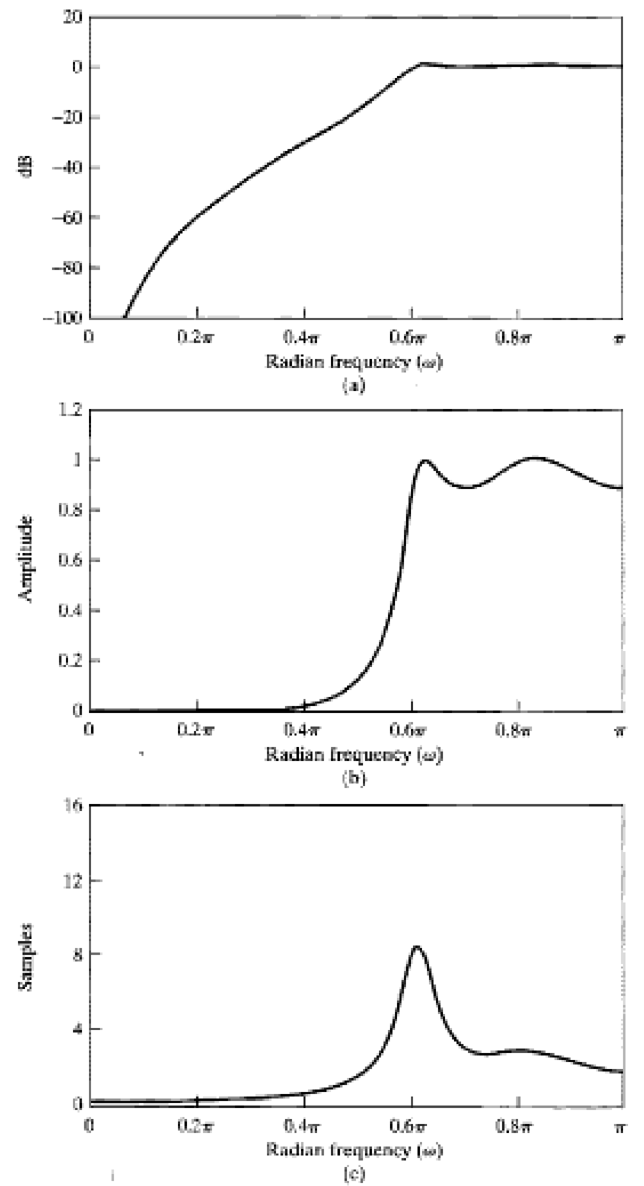


Figure 7.26 Frequency response of 4^{th}-order Chebyshev highpass filter obtained by frequency transformation. (a) Log magnitude in dB. (b) Magnitude. (c) Group delay.

7.5 طراحی فیلترهای FIR :

چندجمله‌ای

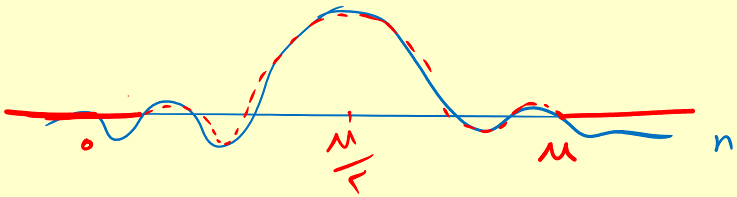
۱. Windowing پنجره‌گذاری

۲. تقریب بهینه

Windowing : پنجره‌گذاری

پاسخ ضربه محدود

پاسخ ضربه ایده‌آل $h_d[n]$



با ضرب کردن در یک پنجره پاسخ ضربه محدود می‌شود.

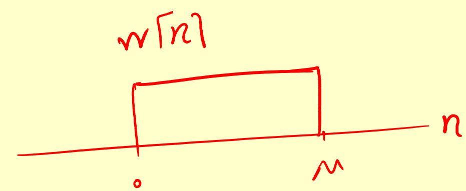
Filter Design by Windowing

- Simplest way of designing FIR filters
- Method is all discrete-time no continuous-time involved
- Start with ideal frequency response

$$H_d\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n} \qquad h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

- Choose ideal frequency response as desired response
- Most ideal impulse responses are of infinite length
- The easiest way to obtain a causal FIR filter from ideal is

$$h[n] = \begin{cases} h_d[n] & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$



- More generally

ضرب

FIR پاسخ

$$h[n] = \underbrace{h_d[n]}_{\text{پاسخ ایده‌آل}}\,\underbrace{w[n]}_{\text{پنجره}} \qquad \text{where} \qquad w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

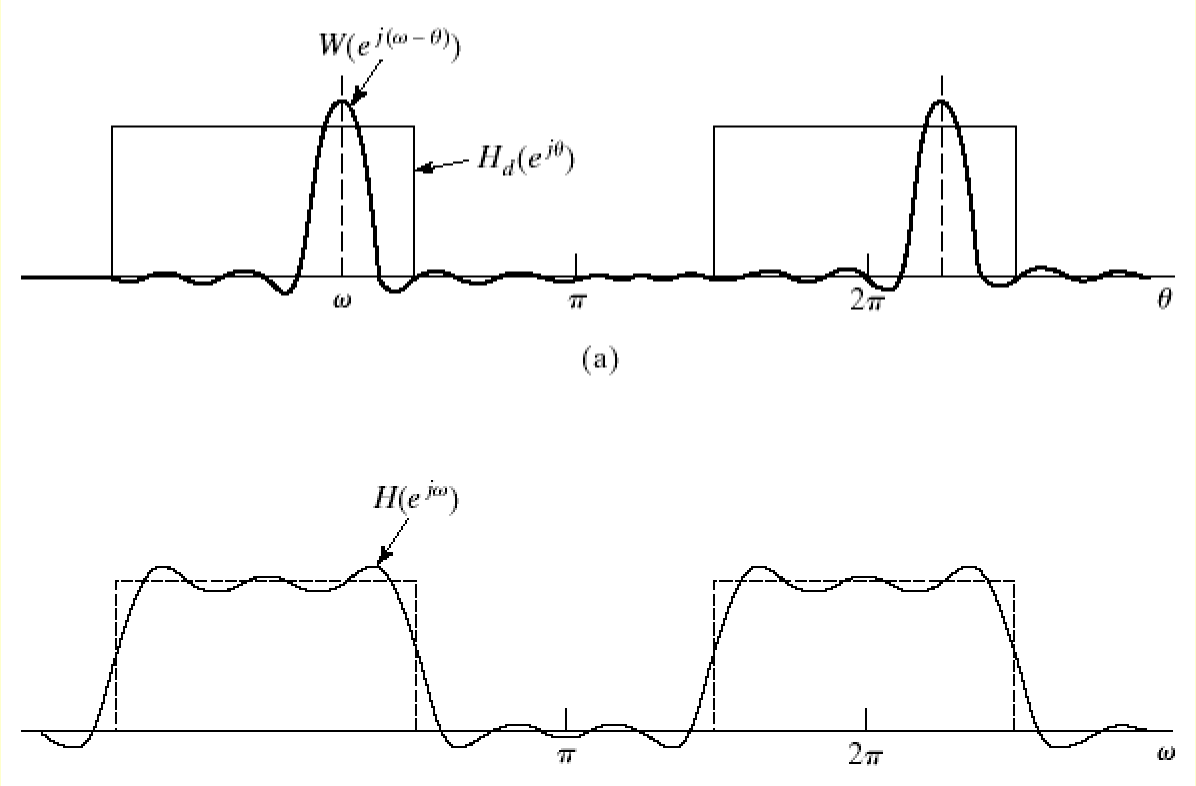
M+1 بطول

Windowing in Frequency Domain

- Windowed frequency response ضرب زمان ↔ فرکانس convolution

$$H\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_d\left(e^{j\omega}\right)W\left(e^{j(\omega-\theta)}\right)d\theta$$

- The windowed version is smeared version of desired response



- If w[n]=1 for all n, then $W(e^{j\omega})$ is pulse train with 2π period

Properties of Windows

$\begin{cases} \Delta\omega_m \\ PSL \end{cases}$

- Prefer windows that concentrate around DC in frequency
  - Less smearing, closer approximation
- Prefer window that has minimal span in time
  - Less coefficient in designed filter, computationally efficient
- So we want concentration in time and in frequency
  - Contradictory requirements
- Example: Rectangular window

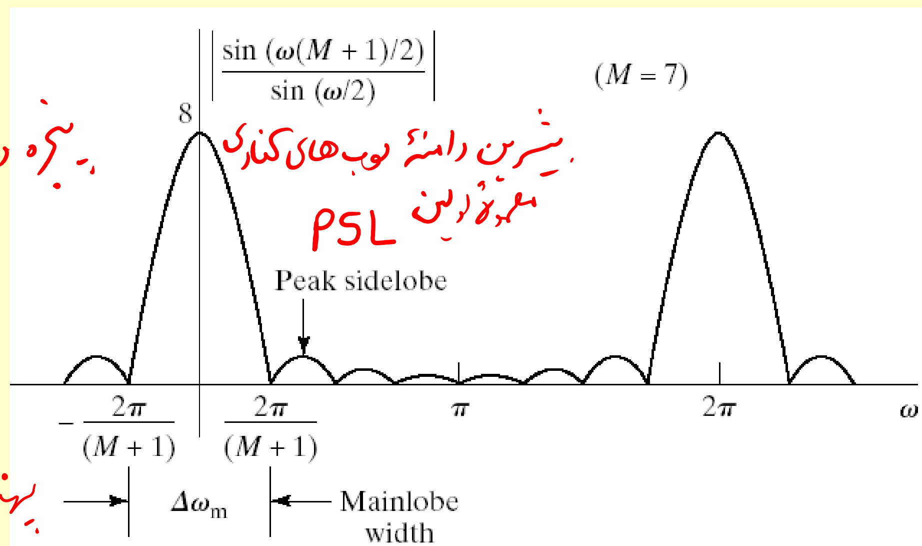
پنجرهٔ مستطیلی:

$$W\left(e^{j\omega}\right) = \sum_{n=0}^{M} e^{-j\omega n} = \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} = e^{-j\omega M/2} \frac{\sin[\omega(M+1)/2]}{\sin[\omega/2]}$$

- Demo

$W(e^{j\omega})$

پنجره در حوزهٔ فرکانس



بیشترین دامنهٔ لوب‌های کناری

معروف به PSL

$\Delta\omega_m$

پهنای لوب اصلی

Rectangular Window

- Narrowest main lob

$\Delta\omega_M$ = 4π/(M+1)

  - Sharpest transitions at discontinuities in frequency

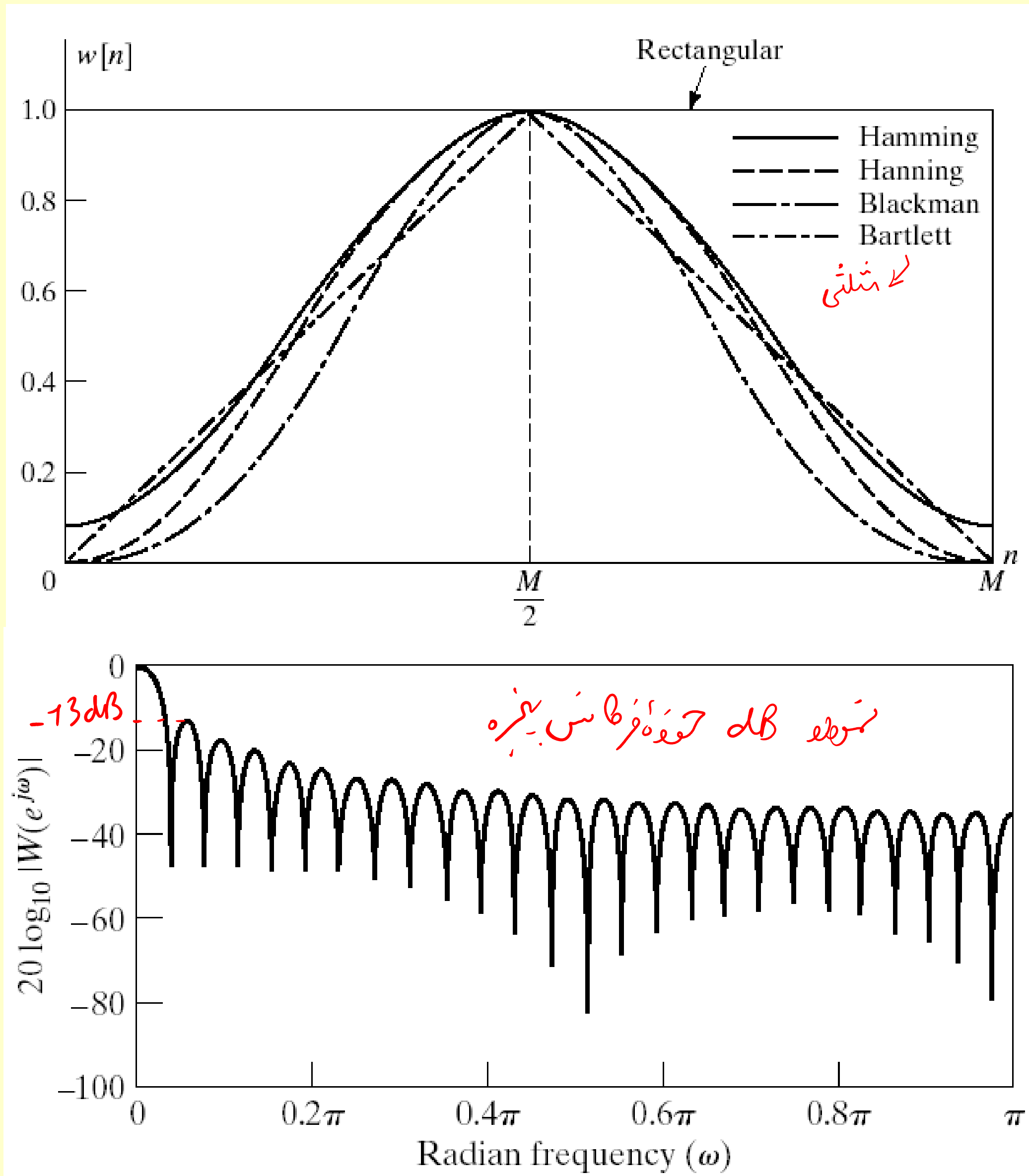
- Large side lobs

PSL = -13 dB

  - Large oscillation around discontinuities

- Simplest window possible

$$w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$



مثلثی Bartlett (Triangular) Window

- Medium main lob

$\Delta\omega_m =$ $8\pi/M$ $\Delta\omega_M$

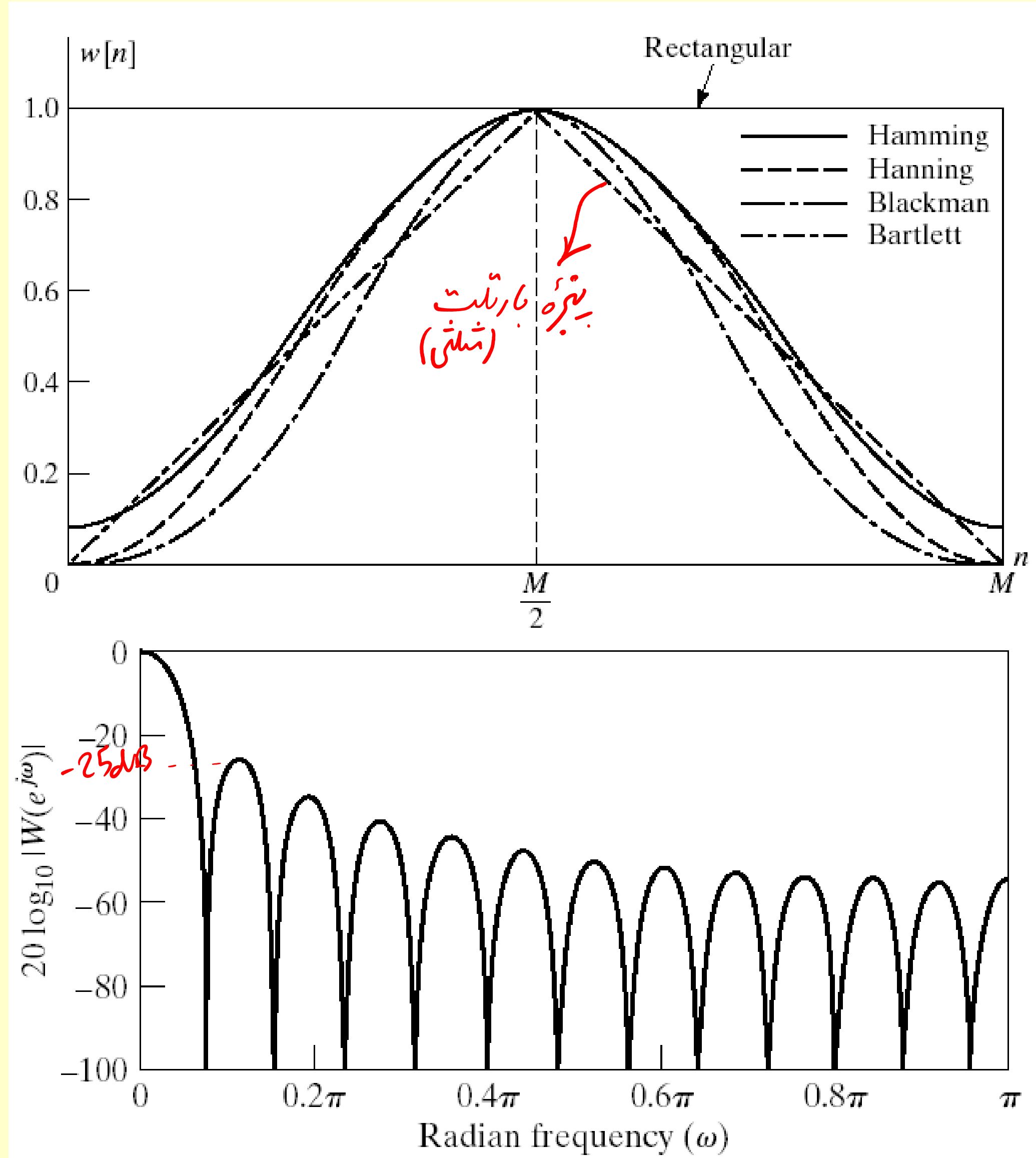
- Side lobs

PSL = -25 dB PSL

- Hamming window performs better

- Simple equation

$$w[n] = \begin{cases} 2n/M & 0 \le n \le M/2 \\ 2 - 2n/M & M/2 \le n \le M \\ 0 & \text{else} \end{cases}$$



Hanning Window

پنجره هنینگ

- Medium main lob

$\Delta\omega_m =$ $8\pi/M$ $\Delta\omega_m$

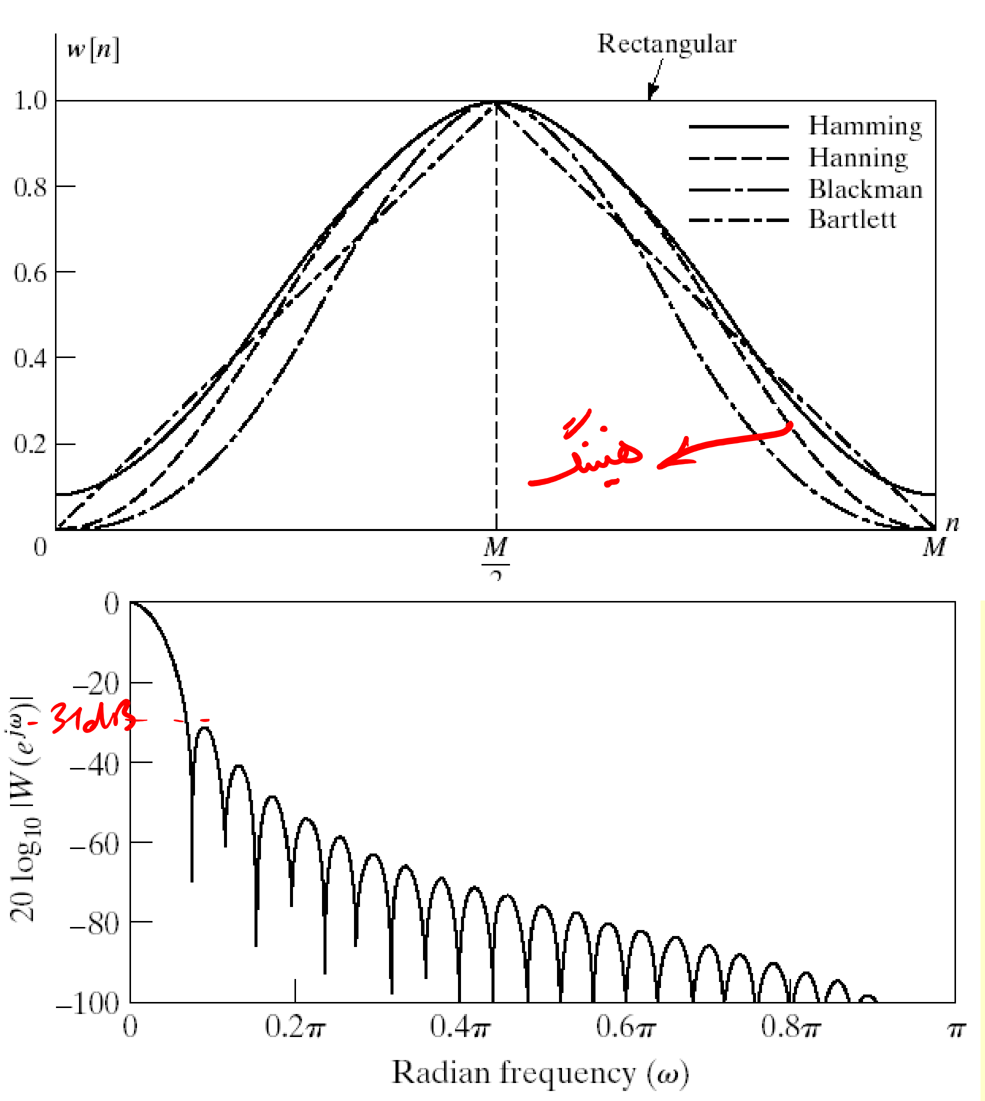
- Side lobs

PSL = -31 dB PSL

- Hamming window performs better

- Same complexity as Hamming

$$w[n] = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M}\right)\right] & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$



Hamming Window پنجرهٔ همینگ

- Medium main lob

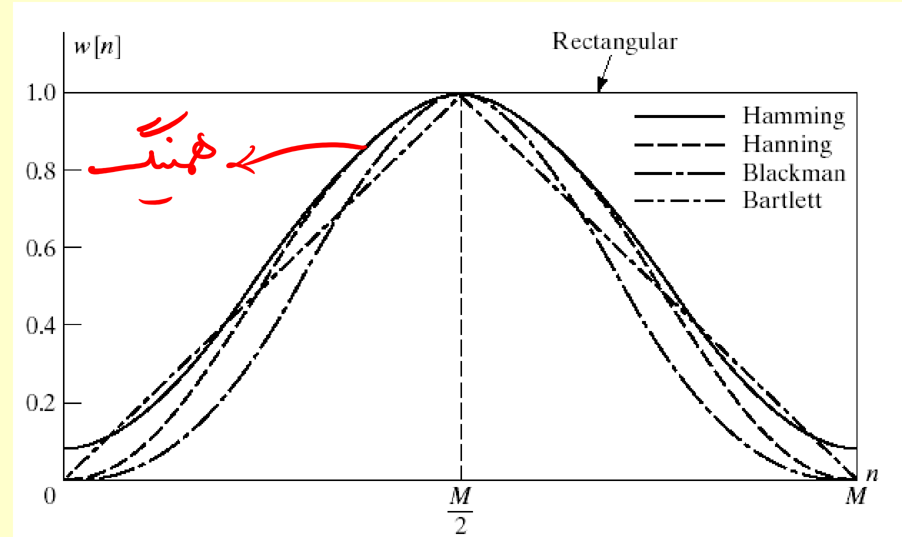
$\Delta\omega_m =$ $8\pi/M$ $\Delta\omega_M$

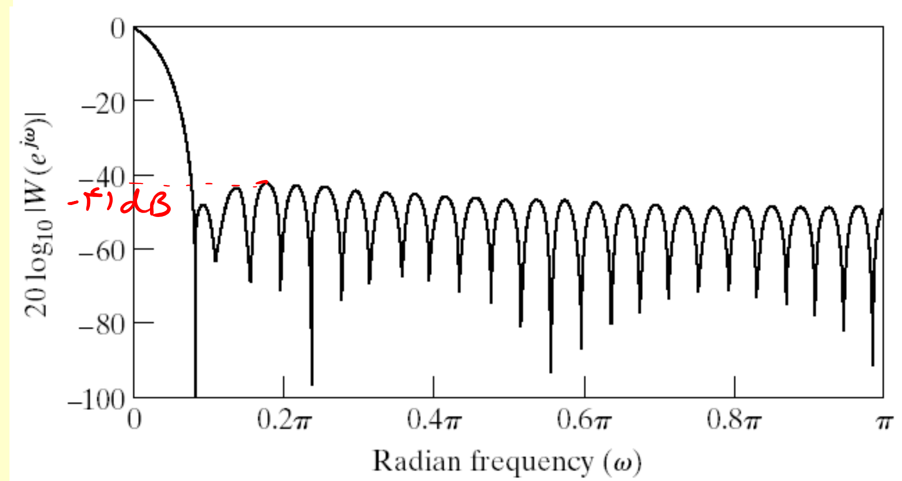
- Good side lobs

PSL = -41 dB PSL

- Simpler than Blackman

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\dfrac{2\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$





Blackman Window

- Large main lob

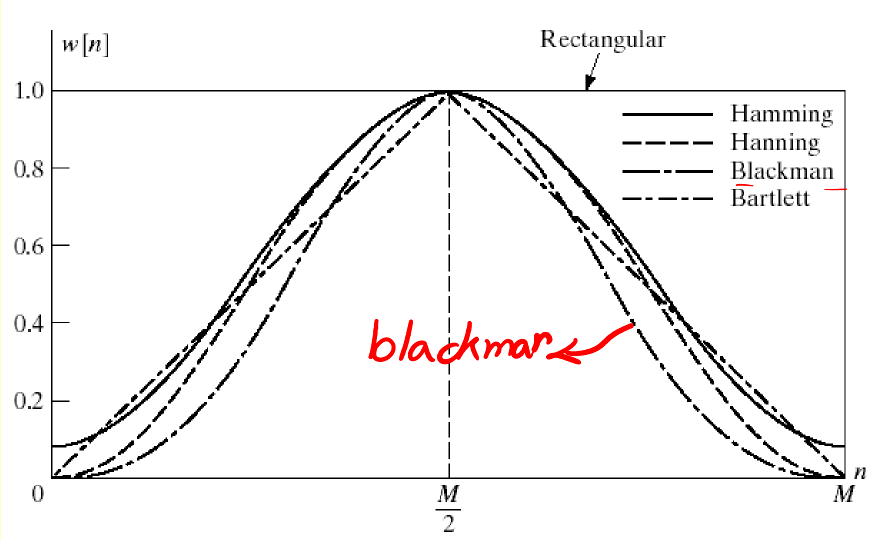
$\Delta\omega_m =$ $12\pi/M$

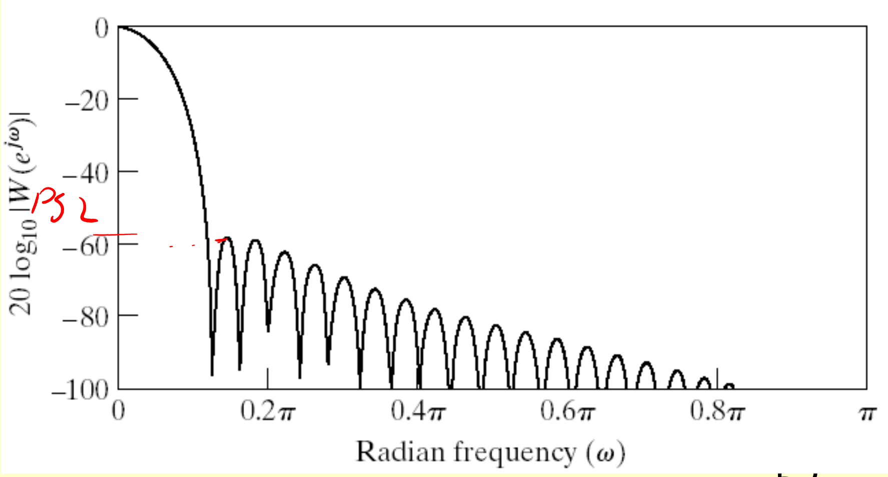
- Very good side lobs

PSL = -57 dB

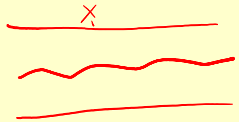
- Complex equation پیچیدگی معادله:

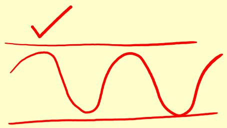
$$w[n] = \begin{cases} 0.42 - 0.5\cos\left(\dfrac{2\pi n}{M}\right) + 0.08\cos\left(\dfrac{4\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$





- [Windows Demo]





حل:

$20\log_{10}\delta = -30\,dB \Rightarrow$ hann, hamming, blackman

$\Delta\omega_M = \frac{\pi}{10} \Rightarrow \frac{8\pi}{M} = \frac{\pi}{10} \Rightarrow M > 80$ Hamming

$\frac{12\pi}{M} = \frac{\pi}{10} \Rightarrow M > 120$ blackman

TABLE 7.2 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	PSL Peak Side-Lobe Amplitude (Relative)	$\Delta\omega_M$ Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB) تولورانس باند عبور و قطع	پنجره بهینه Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hann	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

$-35\,dB \Rightarrow -41\,dB$

* کم و گسسته بودن مقادیر جدول باعث هزینه اضافی میشه.

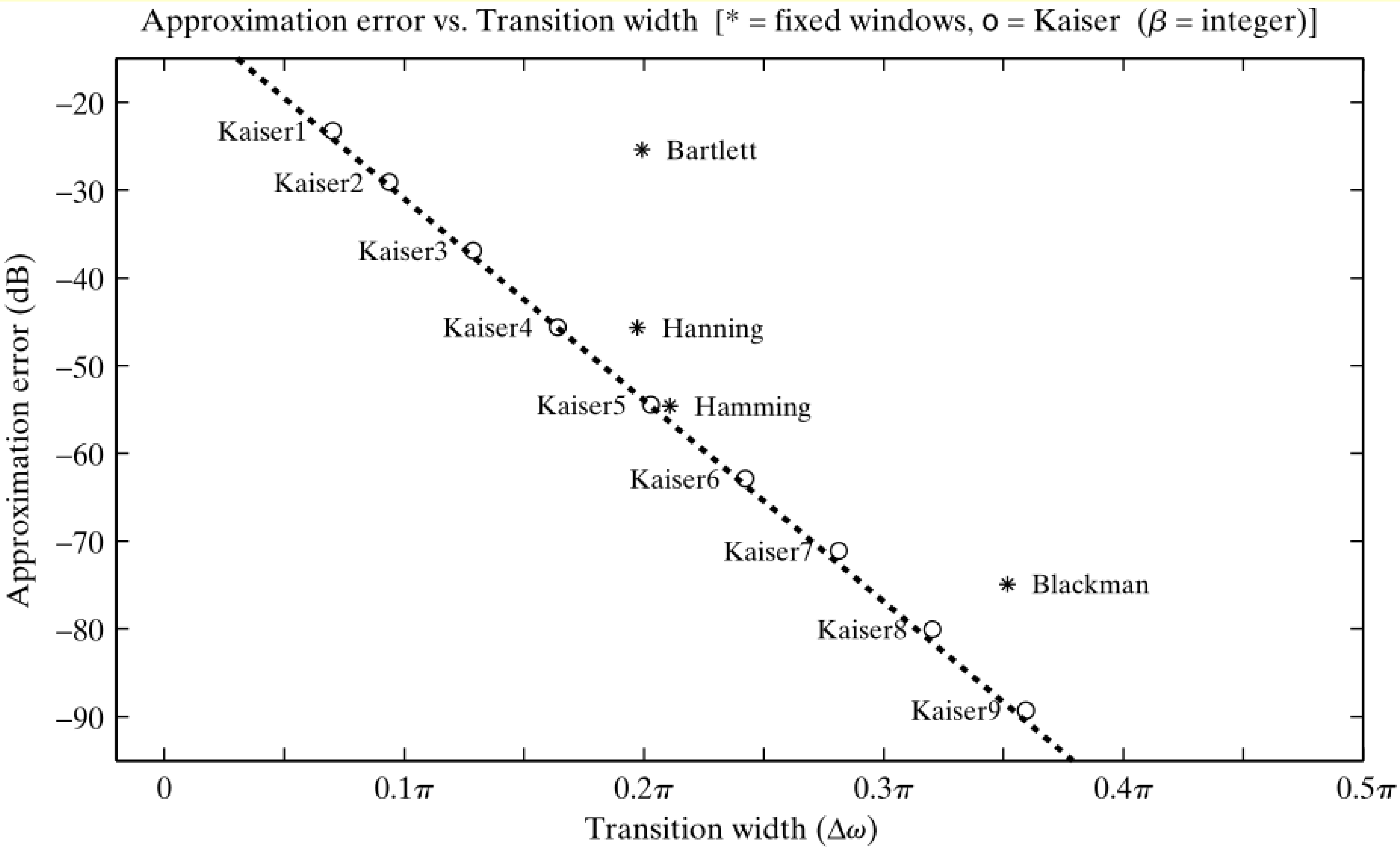


Figure 33 Comparison of fixed windows with Kaiser windows in a lowpass filter design application ($M = 32$ and $\omega_c = \pi/2$). (Note that the designation “Kaiser 6” means Kaiser window with $\beta = 6$, etc.)

Incorporation of Generalized Linear Phase

- Windows are designed with linear phase in mind
  - Symmetric around M/2

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

- So their Fourier transform are of the form

$$W(e^{j\omega}) = W_e(e^{j\omega})e^{-j\omega M/2} \qquad \text{where } W_e(e^{j\omega}) \text{ is a real and even}$$

- Will keep symmetry properties of the desired impulse response
- Assume symmetric desired response

$$H_d(e^{j\omega}) = H_e(e^{j\omega})e^{-j\omega M/2}$$

- With symmetric window

$$A_e(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_e(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

  - Periodic convolution of real functions

Linear-Phase Lowpass filter

- Desired frequency response

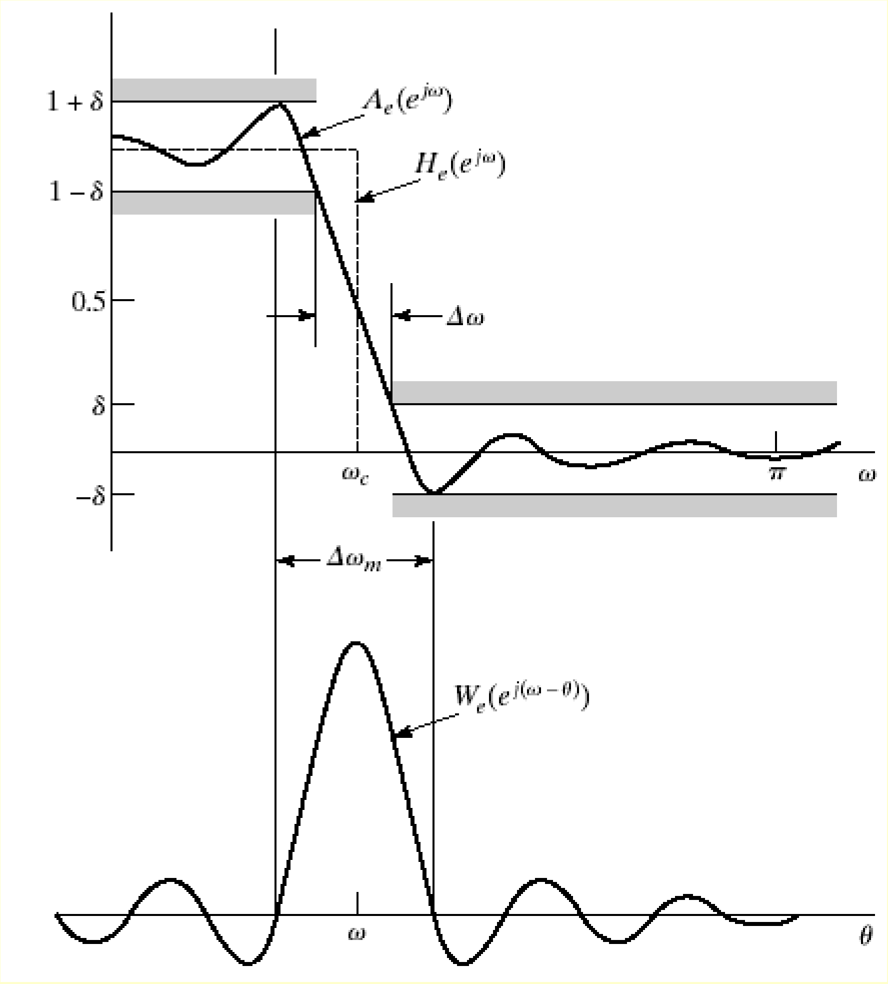
$$H_{lp}\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega M/2} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

- Corresponding impulse response

$$h_{lp}[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)}$$

- Desired response is even symmetric, use symmetric window

$$h[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} w[n]$$



پنجره‌ی بهینه‌ی کایزر

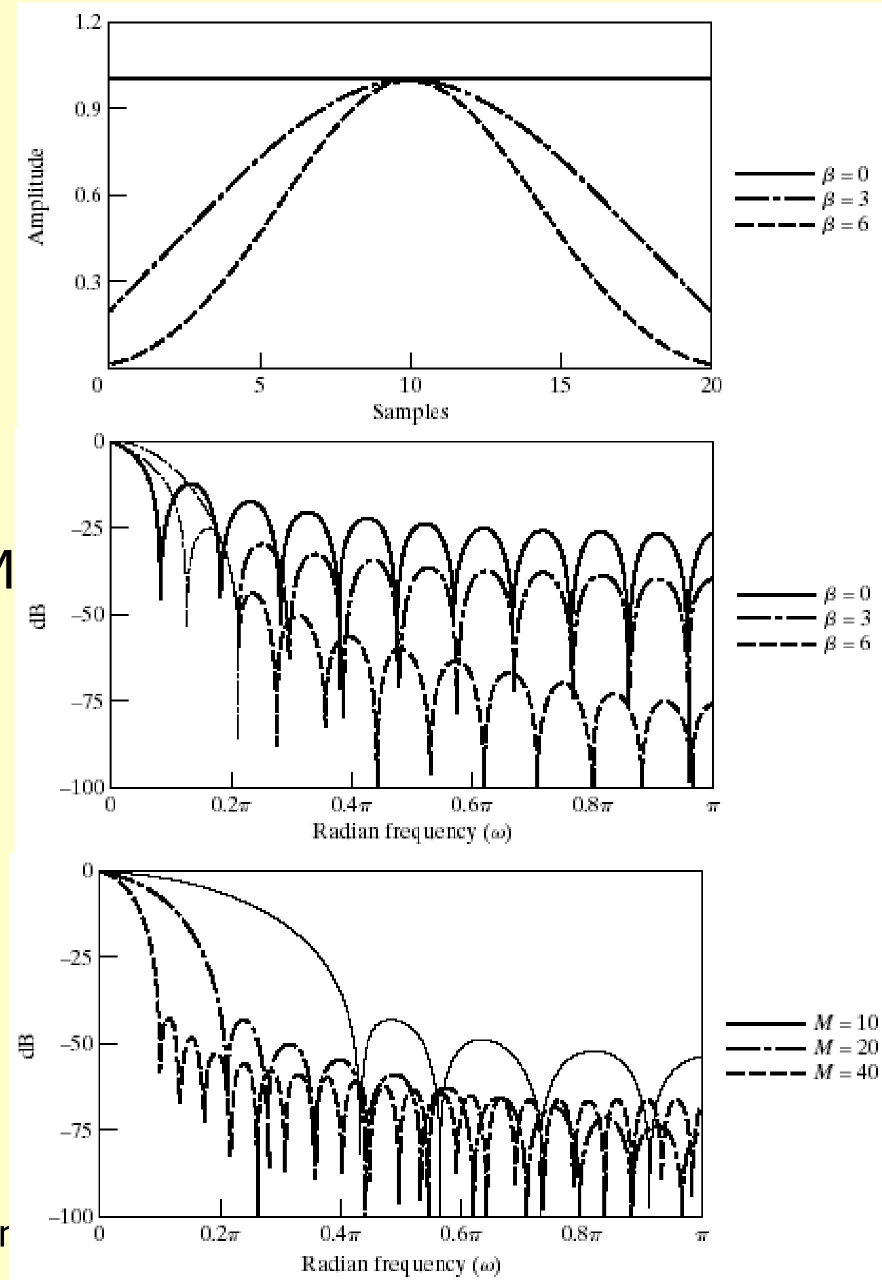
Kaiser Window Filter Design Method

- Parameterized equation forming a set of windows
  - Parameter to change main-lob width and side-lob area trade-off

$$
w[n] = \begin{cases} \dfrac{I_0\left[\beta\sqrt{1-\left(\dfrac{n-M/2}{M/2}\right)^2}\,\right]}{I_0(\beta)} & 0 \le n \le M \\ 0 & \text{else} \end{cases}
$$

  - $I_0(.)$ represents zeroth-order modified Bessel function of 1st kind

I_0 : تابع بسل اصلاح شده‌ی نوع اول



محاسبهٔ پارامترهای بهینهٔ Kaiser از مشخصات فیلتر :

Determining Kaiser Window Parameters

- Given filter specifications Kaiser developed empirical equations
  - Given the peak approximation error δ or in dB as $\boxed{A=-20\log_{10}\delta}$
  - and transition band width $\boxed{\Delta\omega = \omega_s - \omega_p}$
- The shape parameter β should be

$$\beta = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases}$$

- The filter order M is determined approximately by

$$M = \frac{A-8}{2.285\Delta\omega}$$

① الگوریتم طراحی فیلتر FIR به روش Kaiser :

$\omega_s, \omega_p, \delta_2, \delta_1$

Spec
فیلتر

↓

$\delta = \min(\delta_1, \delta_2)$

$A = -20 \log_{10} \delta$

$\Delta\omega = \omega_s - \omega_p$

↓

مثبت A

$$\beta = \begin{cases} 0.1102(A - 8.7) & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 \leq A \leq 50 \\ 0 & A < 21 \end{cases}$$

$$M = \frac{A - 8}{2.285\,\Delta\omega} \pm 2$$

↓

$w(n)$
Kaiser

↓

ideal LPF

$h[n] = h_{id}[n]\, w[n]$

↓

پایان

Example: Kaiser Window Design of a Lowpass Filter

- Specifications $\omega_p = 0.4\pi, \omega_s = 0.6\pi, \delta_1 = 0.01, \delta_2 = 0.001$
- Window design methods assume $\delta_1 = \delta_2 = \underline{0.001} = \delta$ کوچیکترین δ
- Determine cut-off frequency
  - Due to the symmetry we can choose it to be $\underline{\omega_c = 0.5\pi}$
- Compute

$$\underline{\Delta\omega} = \omega_s - \omega_p = 0.2\pi \qquad A = -20\log_{10}\delta = \underline{60}$$

دقت

- And Kaiser window parameters

$$\beta = 5.653 \qquad \boxed{M = 37}$$

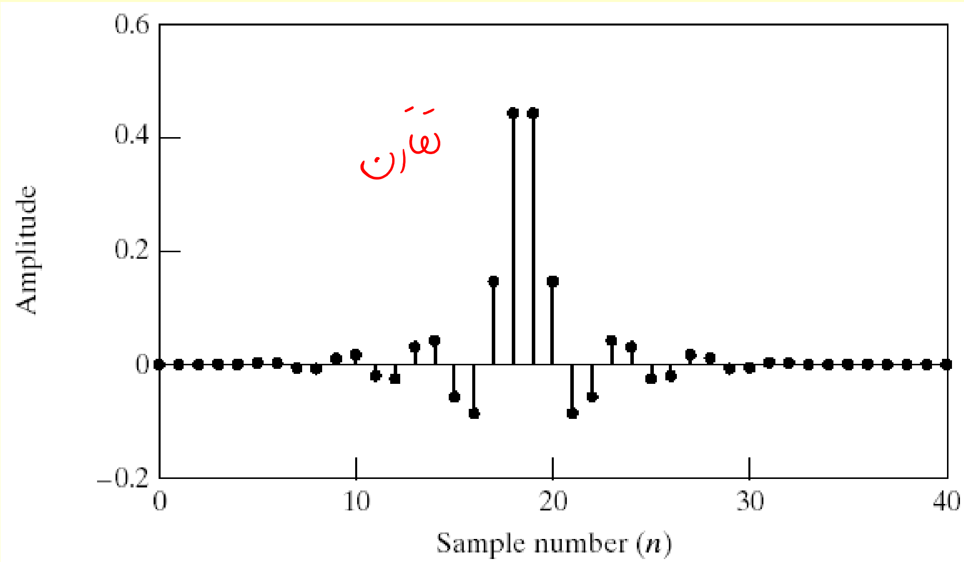
- Then the impulse response is given as w[n]

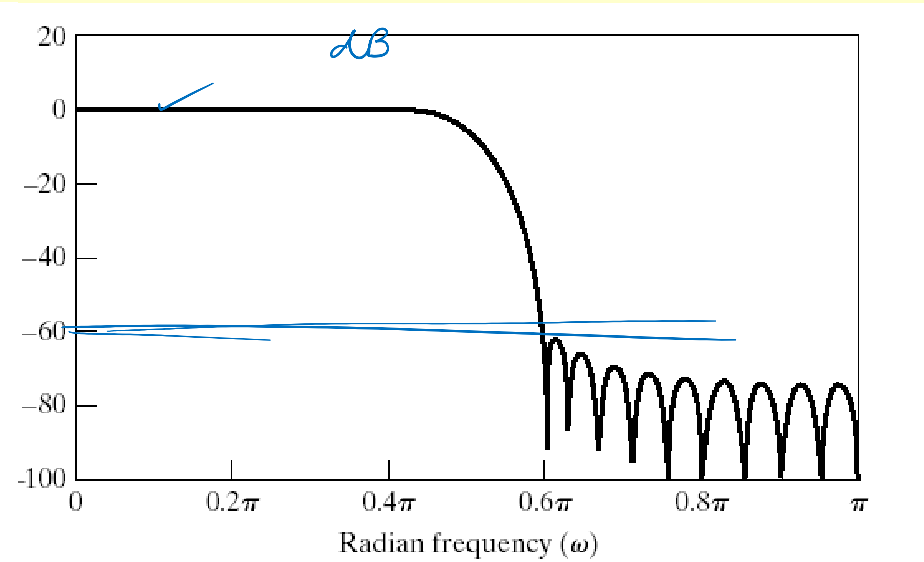
$$h[n] = h_d[n]w[n]$$

kaiser

$$\Rightarrow h[n] = \begin{cases} \underbrace{\dfrac{\sin[0.5\pi(n-18.5)]}{\pi(n-18.5)}}_{h_d[n]} \dfrac{I_0\left[5.653\sqrt{1-\left(\dfrac{n-18.5}{18.5}\right)^2}\right]}{I_0(5.653)} & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

blackman: $\frac{۲}{۱۰}\cancel{\pi} = \frac{۱۲\cancel{\pi}}{M} \Rightarrow M = \frac{۱۲۰}{۲} = 60$

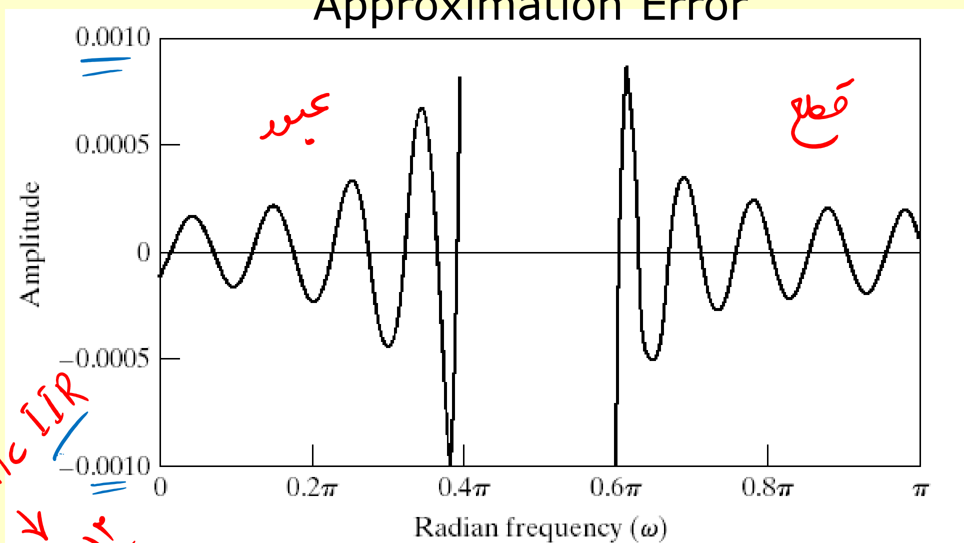




Example Cont'd

Linear

Approximation Error



FIR { فاز خطی امکان
GRD ثابت

به هزینه مرتبه بالاتر
M = 37
37/2 ≈ 19 → مرتبه‌ها
Elliptic IIR → ۱۲: مرتبه‌ش

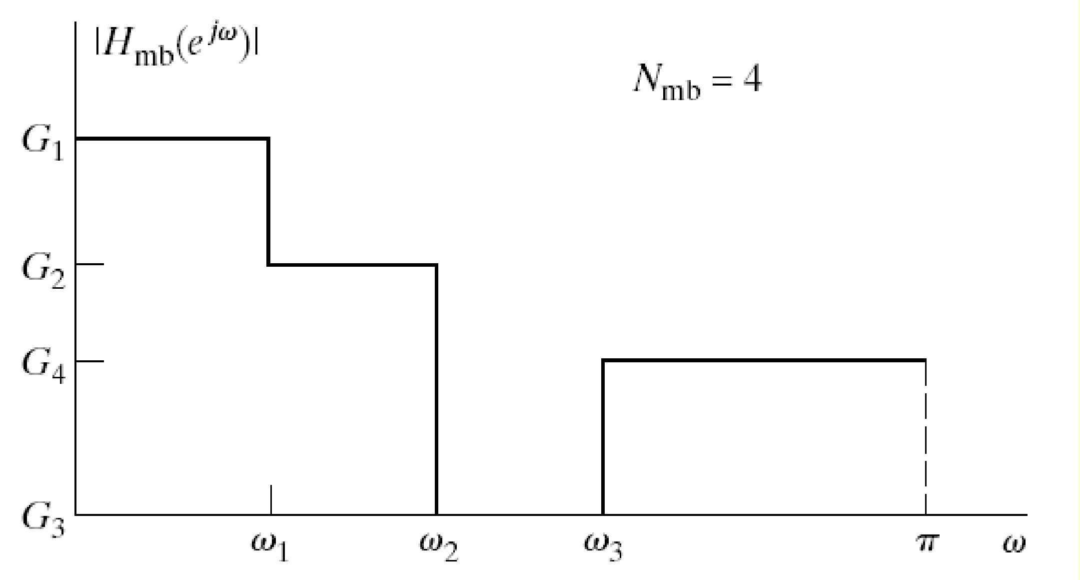
تا پایان پنجره kaiser

General Frequency Selective Filters

- A general multiband impulse response can be written as

$$h_{mb}[n] = \sum_{k=1}^{N_{mb}} \left(G_k - G_{k+1}\right) \frac{\sin \omega_k (n - M/2)}{\pi (n - M/2)}$$

- Window methods can be applied to multiband filters
- Example multiband frequency response
  - Special cases of
    - Bandpass
    - Highpass
    - Bandstop



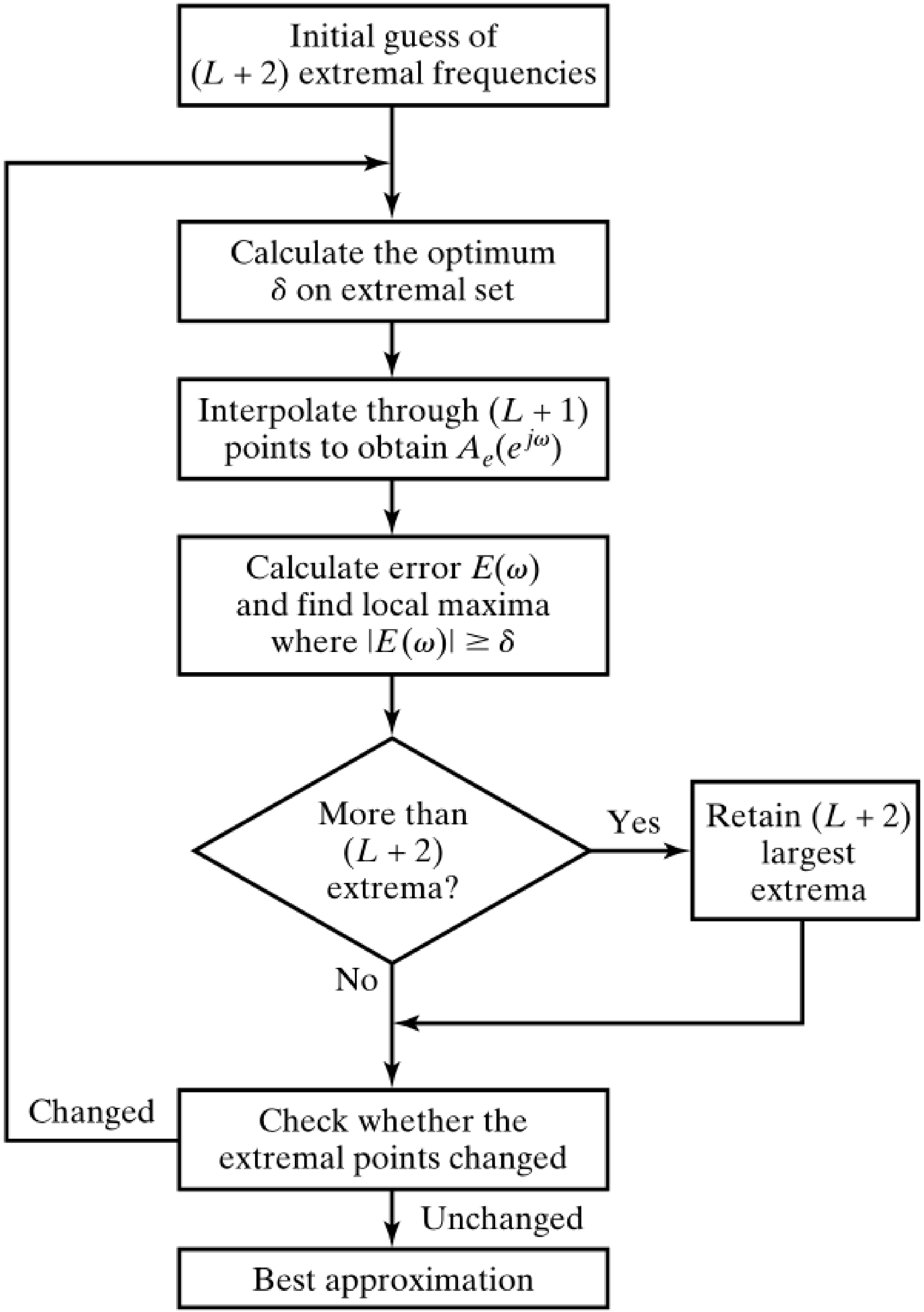


Figure 50 Flowchart of Parks–McClellan algorithm.

Parks-McClellan

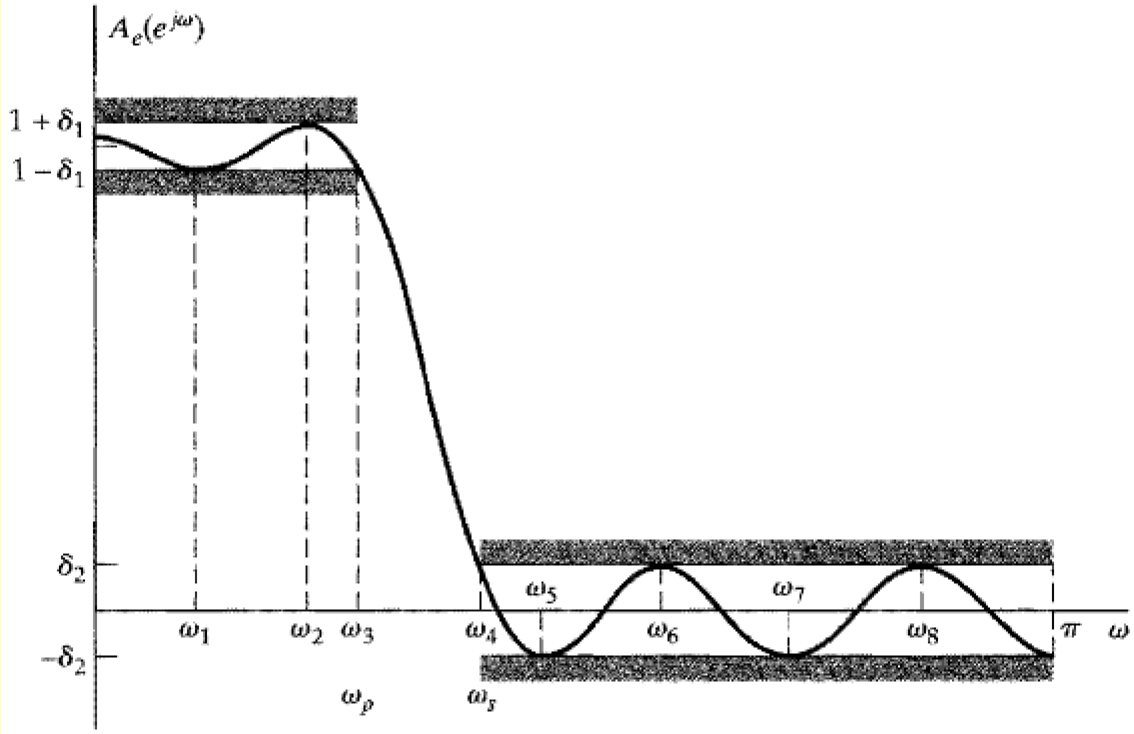


Figure 7.44 Typical example of a lowpass filter approximation that is optimal according to the alternation theorem for $L = 7$.

مثال

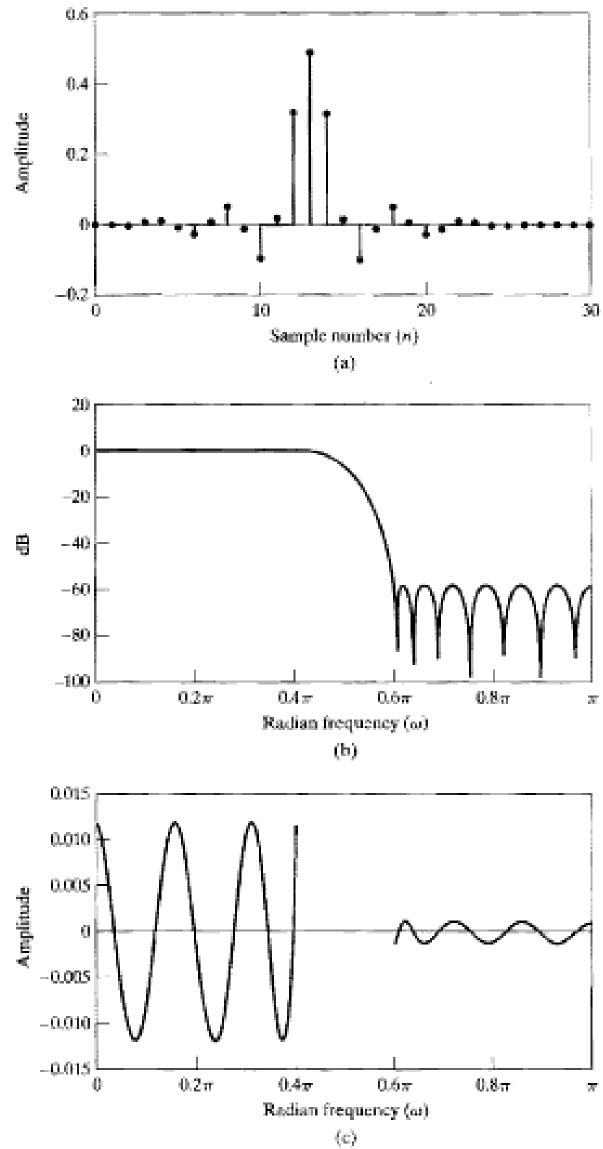


Figure 7.52 Optimum type I FIR lowpass filter for $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $K = 10$, and $M = 26$. (a) Impulse response. (b) Log magnitude of the frequency response. (c) Approximation error (unweighted).

مثال

passband edge frequency ω_p	=	0.22π
stopband edge frequency ω_s	=	0.29π
maximum passband gain	=	0 dB
minimum passband gain	=	−1 dB
maximum stopband gain	=	−40 dB.

TABLE 7.3 ORDERS OF DESIGNED FILTERS.

Filter design	Order
Butterworth	18
Chebyshev I	8
Chebyshev II	8
Elliptic	5
Kaiser	63
Parks–McClellan	44

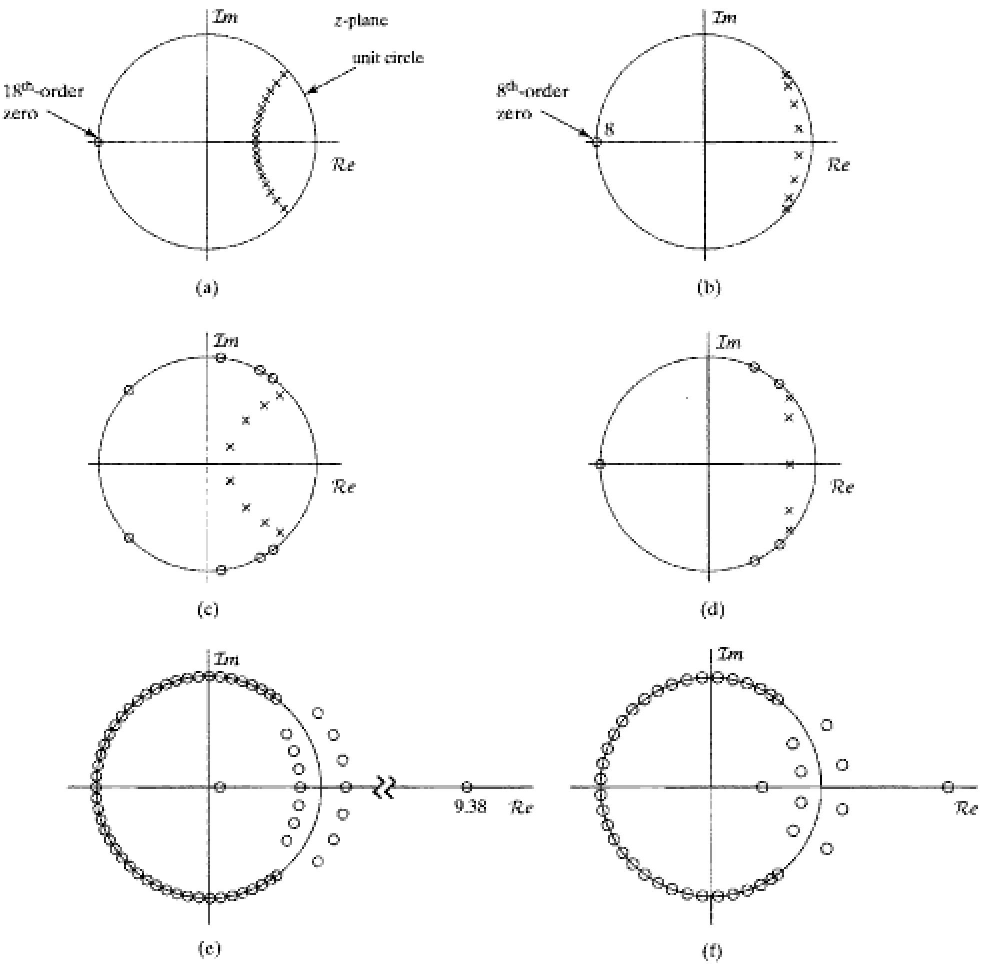


Figure 7.57 Pole–zero plots for the six designs. (a) Butterworth filter. (b) Chebyshev I filter. (c) Chebyshev II filter. (d) Elliptic filter. (e) Kaiser filter. (f) Parks–McClellan filter.