فصل نهم: محاسبهٔ DFT :

الگوریتم های محاسبهٔ سریع DFT : حجم محاسباتی کمتر یا سرعت محاسبهٔ بالاتر

computational complexity

Burden

- ✓ Goertzel
- Winograd
- ✓ chirp

→ مبتنی بر کانولوشن

→ الگوریتم محاسبهٔ DFT

- → Decimation in Time
- → Decimation in Frequency

→ ساختارهای FFT ←

Fast Fourier Transform

تبدیل فوریهٔ سریع

هدف :

$x[n]$ داریم (N تا مولفه) ← $X[k]$ را به دست آوریم. (M تا مولفهٔ)

9.1: محاسبهٔ مستقیم DFT:

رابطهٔ آنالیز

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}}\ , \quad k = 0, 1, \dots, N-1$$

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

رابطهٔ سنتز

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-nk} \ ; \quad n = 0, 1, \dots, N-1$$

نکته: حجم محاسباتی IDFT نیز با حجم محاسبهٔ DFT برابر است $(O(N^2))$

$$X[k] = \underbrace{x[0]\,W_N^{0}}_{\text{ضرب مختلط}} + \underbrace{x[1]\,W_N^{k}}_{\text{ضرب مختلط}} + x[2]\,W_N^{2k} + \dots + x[N-1]\,W_N^{k(N-1)}$$

جمع مختلط

$W_N^0 = 1$
نیاز به ضرب ندارد

N تا ضرب مختلط
$N-1$ تا جمع مختلط
برای هر $X[k]$

$$\underbrace{2N + 2N - 2}_{4N-2}$$

هر ضرب مختلط: ۴ ضرب حقیقی، ۲ جمع حقیقی

$$z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2)$$
$$= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$$

هر جمع مختلط ← ۲ جمع حقیقی

$$\overbrace{x_1 + jy_1 + x_2 + jy_2}^{+}$$

$4N^2$ ضرب حقیقی
$N(4N-2)$ جمع حقیقی

N^2 ضرب مختلط
$N(N-1)$ جمع مختلط
برای کل $X[k]$ ها
N تا

معیار مقایسه: از مرتبهٔ N^2 $\quad O(N^2)$ order

9.2 الگوریتم FFT روش Decimation in Time :

فرض: $N = 2^{v}$

(برای مثال $N = 2, 4, 8, 16, 32, \dots$)

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

$$= \sum_{\substack{n=0 \\ n \text{ زوج}}}^{N-1} x[n] W_N^{nk} + \sum_{\substack{n=0 \\ n \text{ فرد}}}^{N-1} x[n] W_N^{nk}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r] W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r] \left(W_N^{2}\right)^{rk} + W_N^{k} \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] \left(W_N^{2}\right)^{rk}$$

$$= \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r] W_{\frac{N}{2}}^{rk}}_{\text{DFT . } \frac{N}{2} \text{ نقطه‌ای از مولفه‌های زوج } x[n]} + W_N^{k} \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r+1] W_{\frac{N}{2}}^{rk}}_{\text{DFT . } \frac{N}{2} \text{ نقطه‌ای از مولفه‌های فرد } x[n]} = G[k] + W_N^{k} H[k]$$

$N = 4$

$n = 0, 1, 2, 3$

زوج فرد

$n \rightarrow 2r$

$r = 0, \dots, \frac{N}{2} - 1$

$r = 0, 1$

$1, 3$

$$W_N^{2} = \left(e^{-j\left(\frac{2\pi}{N}\right)}\right)^{2} = e^{-j\frac{2\pi}{N/2}} = W_{\frac{N}{2}}$$

$\frac{N}{2}$ ها: دلیل 2^{v} انتخاب کردن N

Figure 9.3 Illustration of the basic principle of decimation-in-time.

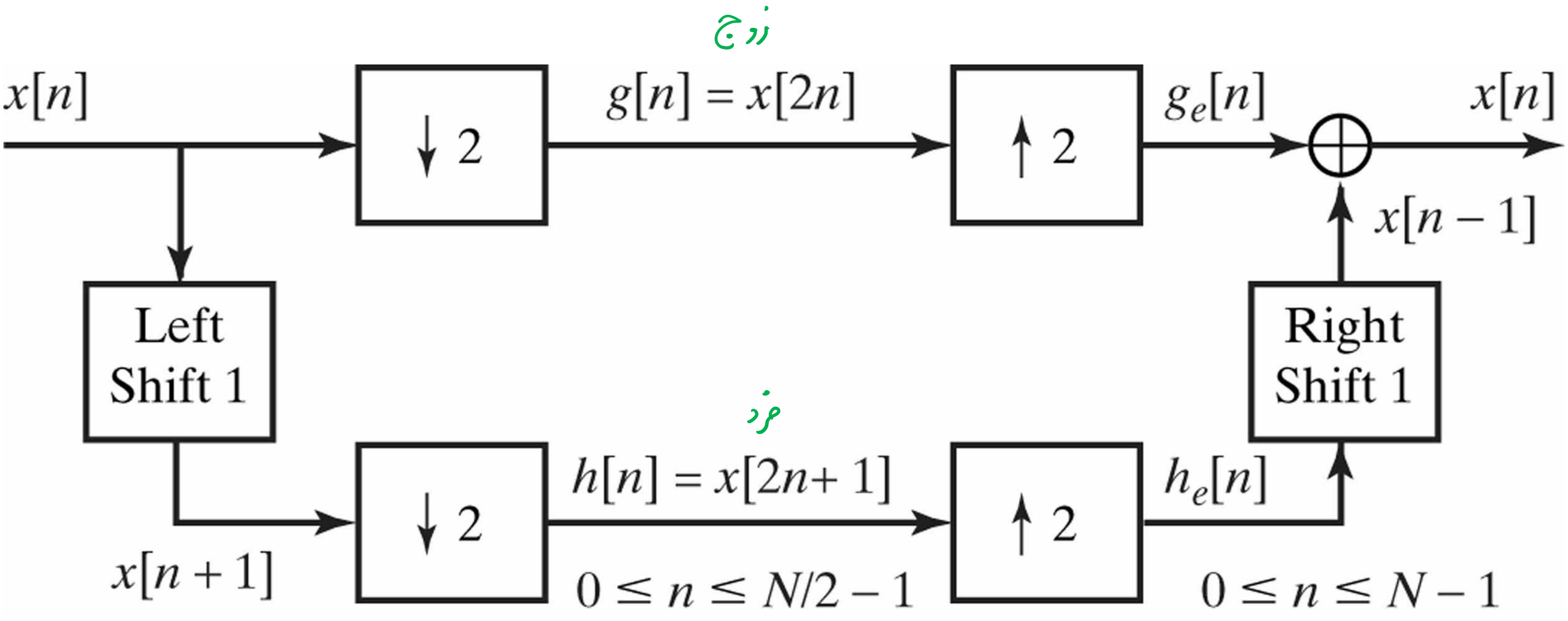


Figure 9.4 Flow graph of the decimation-in-time decomposition of an N-point DFT computation into two $(N/2)$-point DFT computations ($N = 8$).

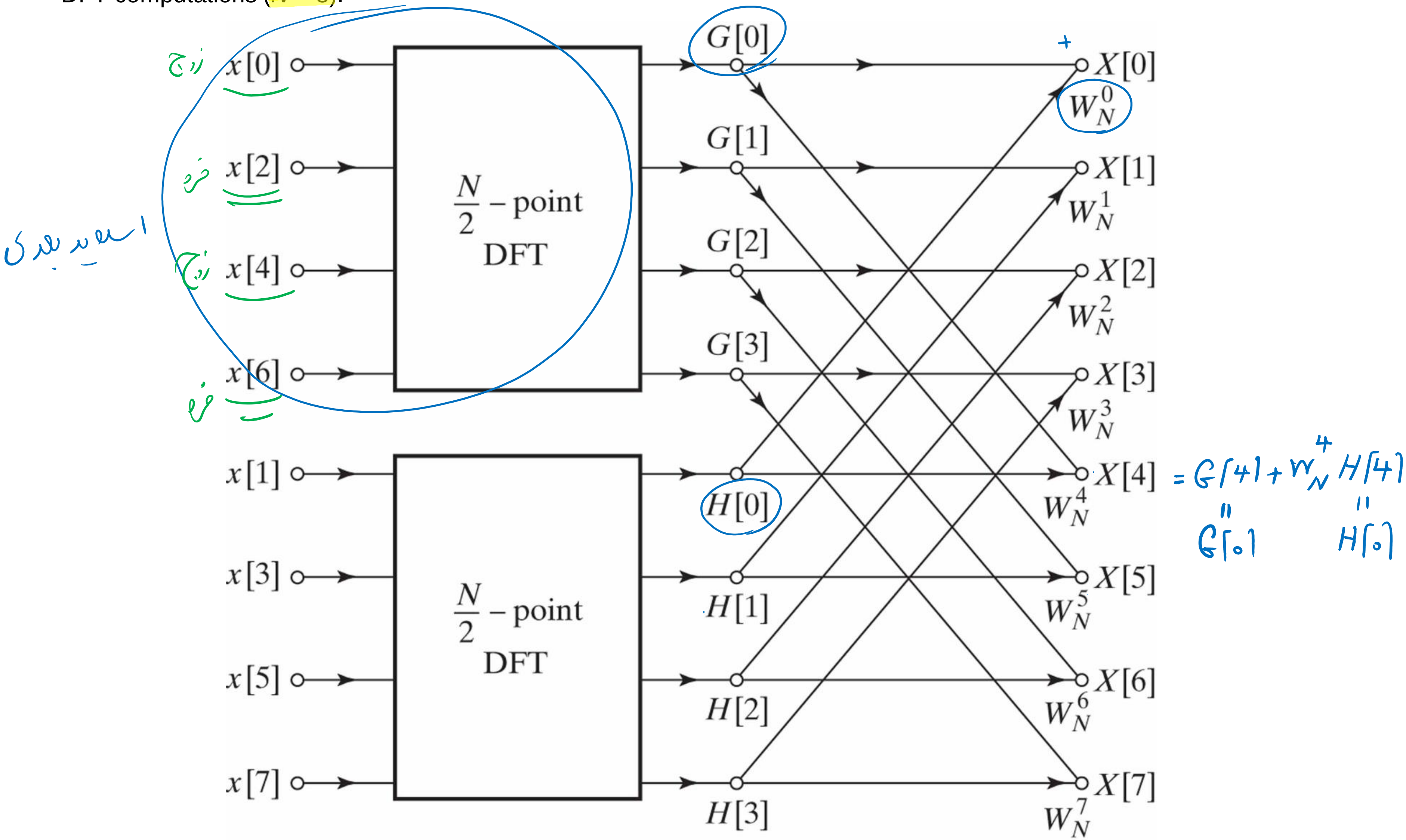


Figure 9.5 Flow graph of the decimation-in-time decomposition of an ($N/2$)-point DFT computation into two ($N/4$)-point DFT computations ($N = 8$).

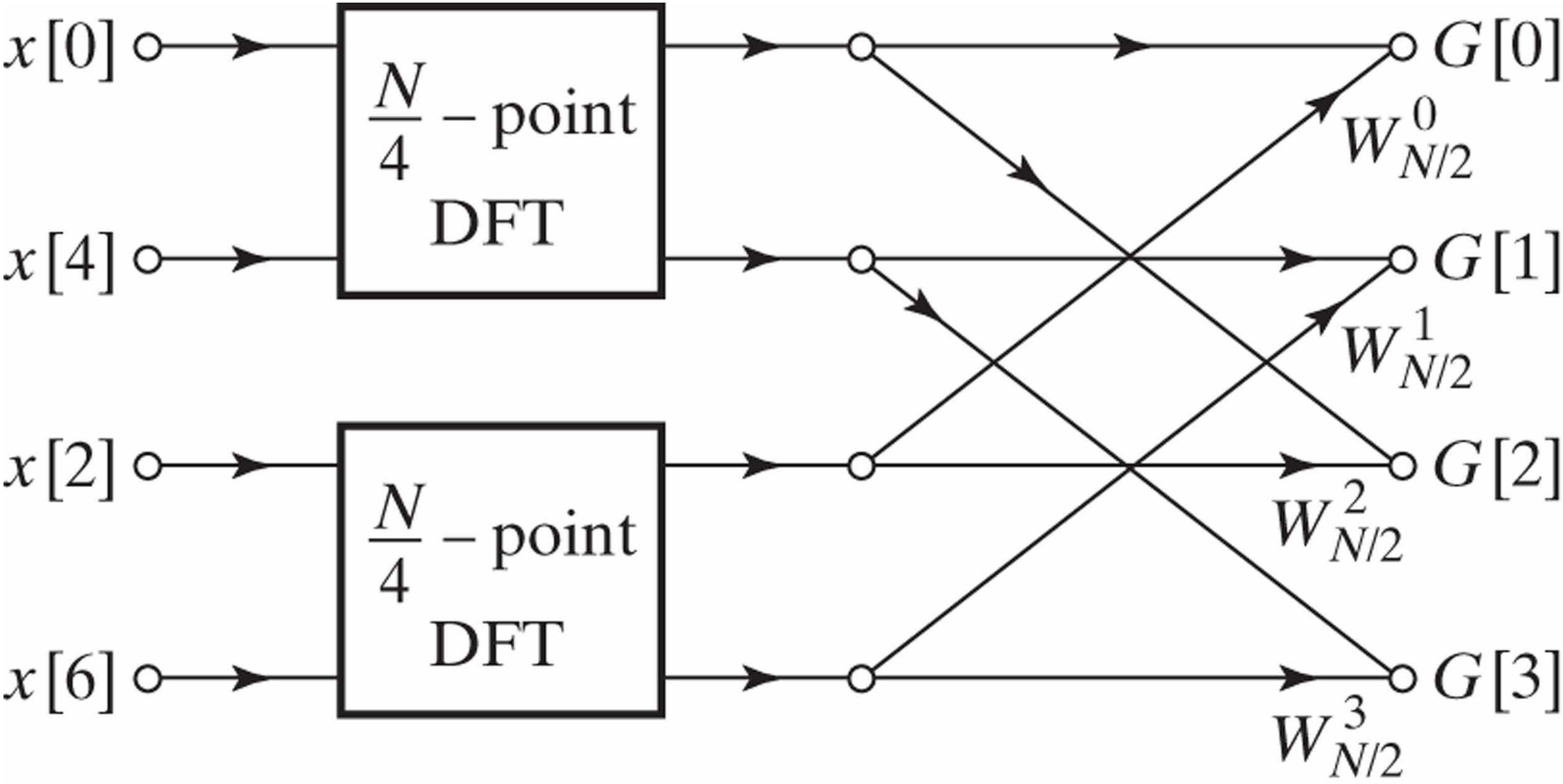


Figure 9.6 Result of substituting the structure of Figure 9.5 into Figure 9.4.

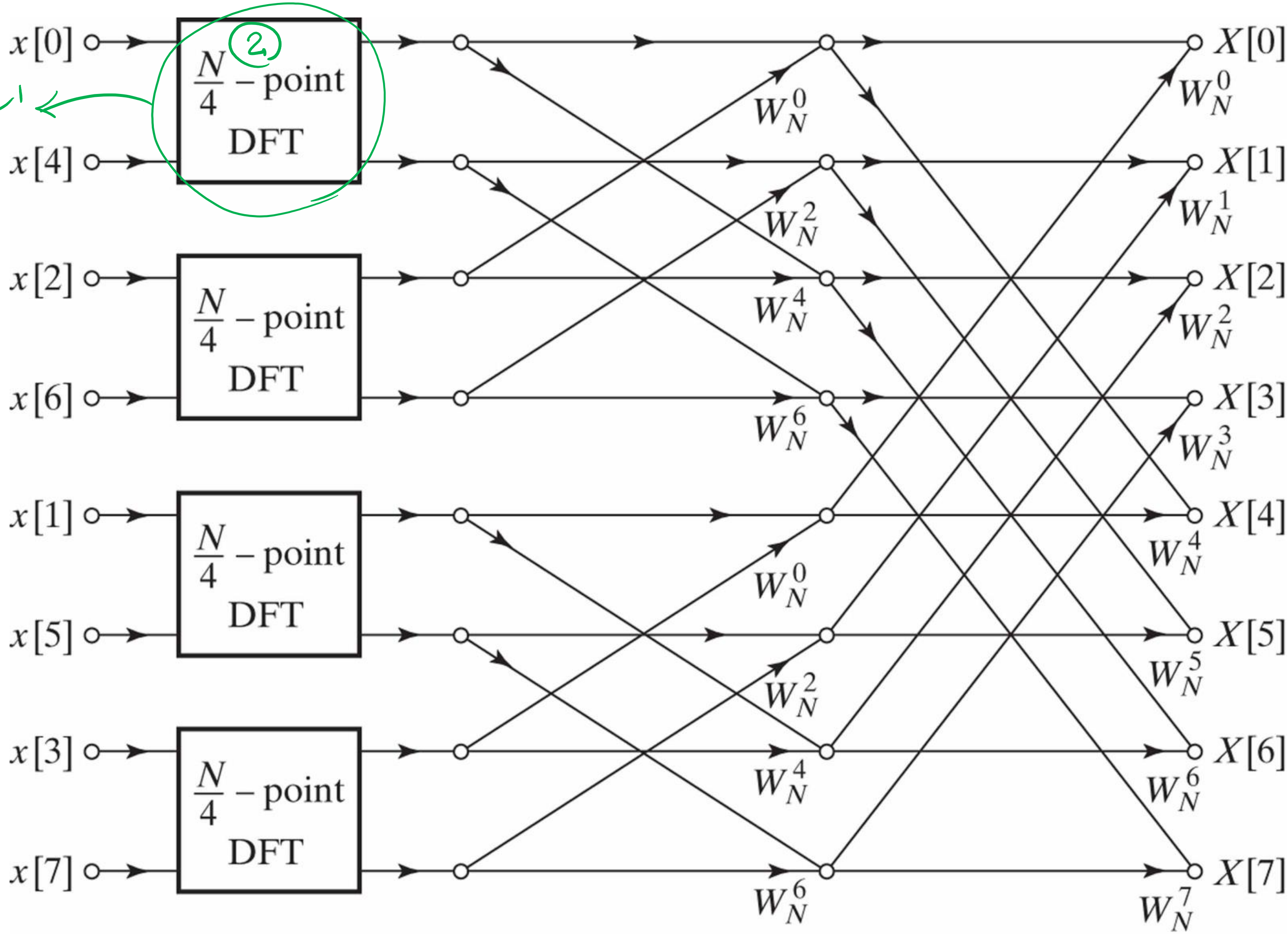


Figure 9.7 Flow graph of a 2-point DFT.

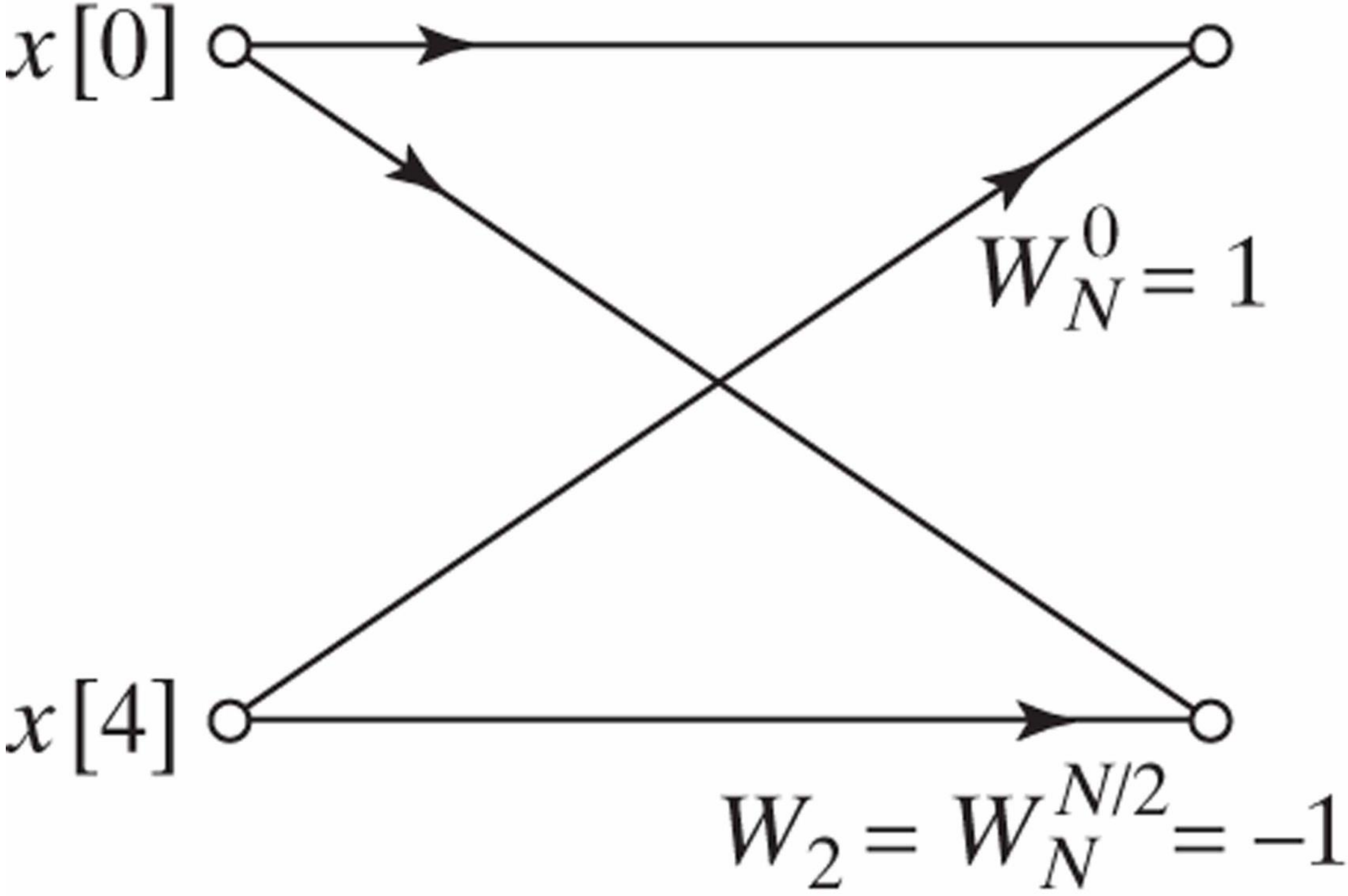
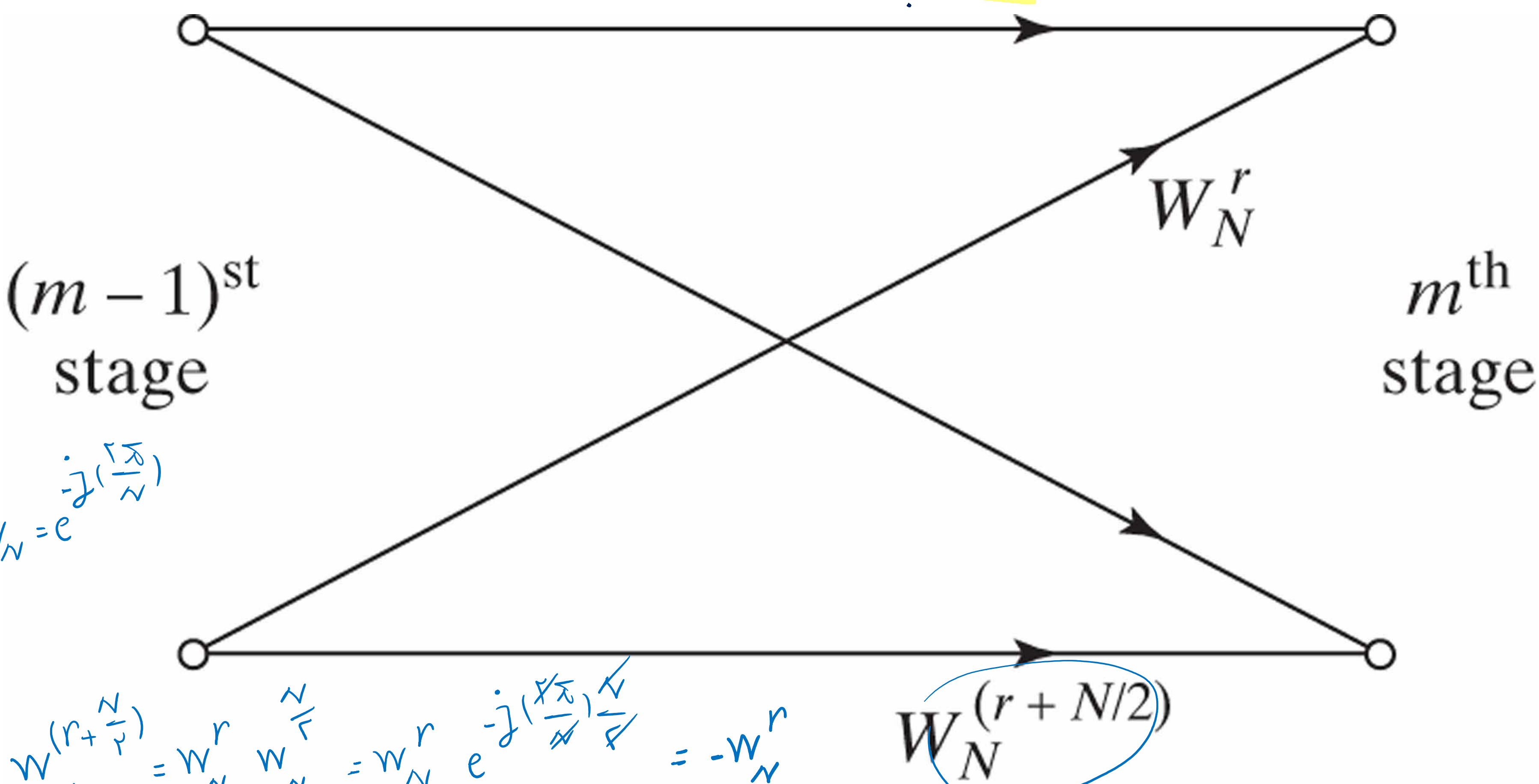


Figure 9.8 Flow graph of basic butterfly computation in Figure 9.9.

ساختار پروانه ای :



$W_N = e^{-j(\frac{2\pi}{N})}$

$W_N^{(r+\frac{N}{2})} = W_N^r \, W_N^{\frac{N}{2}} = W_N^r \, \underbrace{e^{-j(\frac{2\pi}{\cancel{N}})\frac{\cancel{N}}{2}}}_{-1} = -W_N^r$

slide Fig 9.10

تعداد ضرب های هر گام : ۴ ضرب یا $\frac{N}{2}$ ضرب
جمع های هر گام : ۸ جمع یا N جمع

$2^{\nu} = N = 8 = 2^{3} \longrightarrow$ تعداد گام ها $\nu = \log_2 N$

Figure 9.9 Flow graph of complete decimation-in-time decomposition of an 8-point DFT computation.

تعداد ضرب: $\frac{N}{2}\log_2 N$

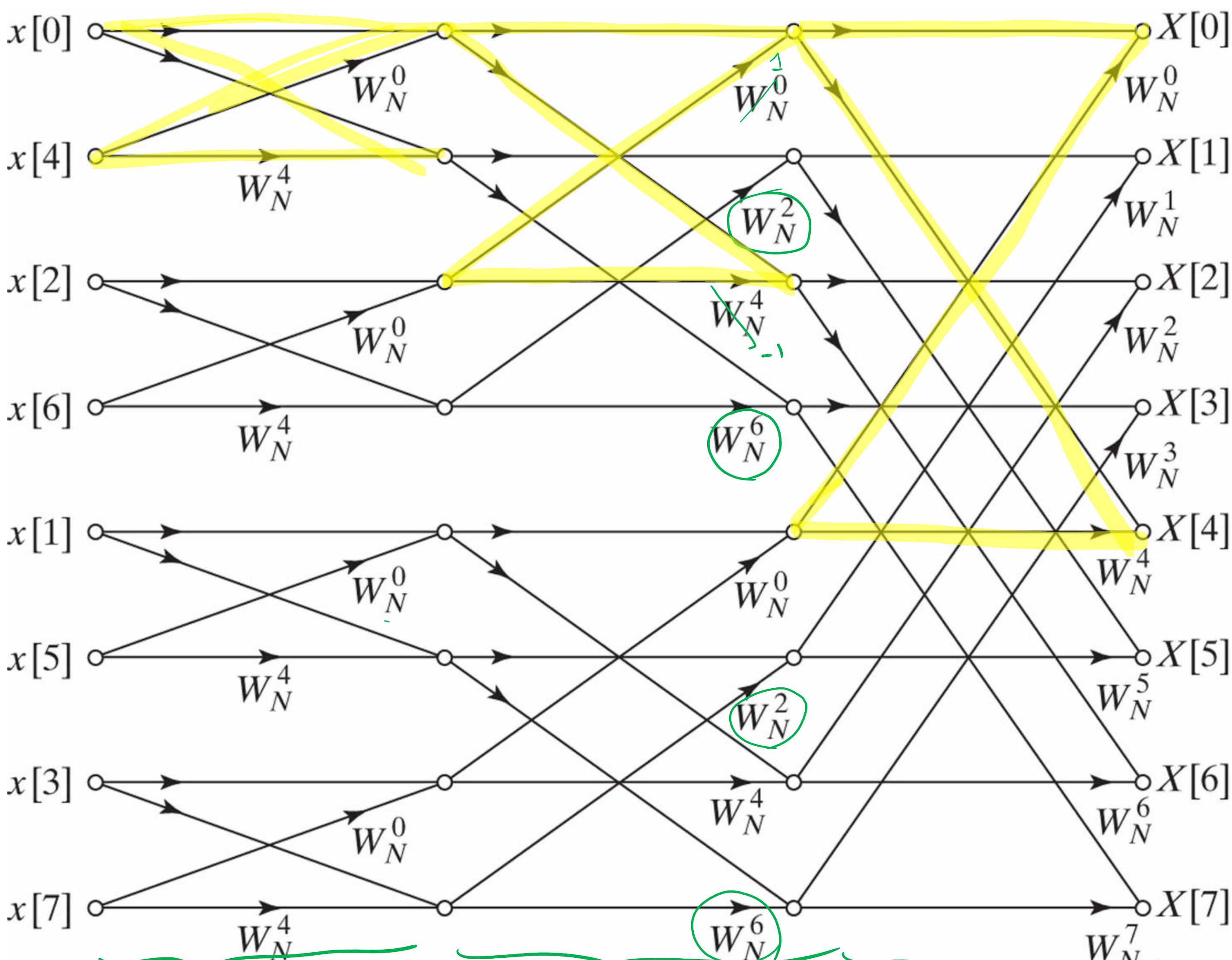
جمع: $N\log_2 N$

حقیقی:

ضرب $2N\log_2 N$

جمع $3N\log_2 N$

$O(N\log_2 N)$



گام اول   گام دوم   گام سوم

مقایسهٔ پیچیدگی محاسباتی DFT به روش مستقیم و FFT :

DFT از مرتبهٔ N^2

FFT از مرتبهٔ $N\log_2 N$

مثال:

$$N = 2^{10} = 1024$$

$$\text{DFT}: \quad N^2 = 2^{20} = 1{,}048{,}576$$

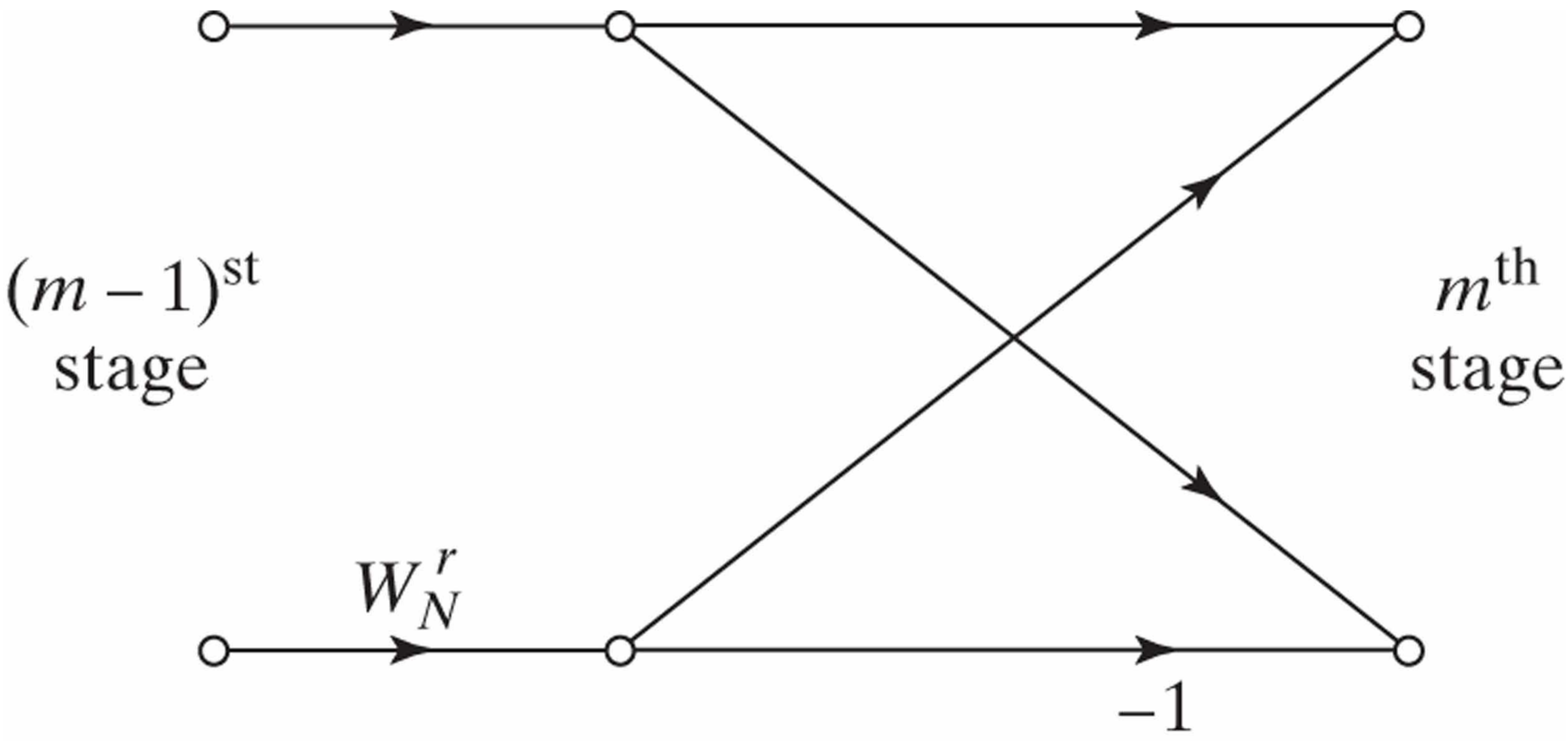
$$\text{FFT}: \quad N\log_2 N = 2^{10} \times 10 = 10{,}240$$

$$\frac{2^{20}}{10 \times 2^{10}} = \underline{100}$$

fft در MATLAB

Figure 9.10 Flow graph of simplified butterfly computation requiring only one complex multiplication.

ساده شدهٔ ساختار پروانه ای از Fig 9.8 :



ثبت درمحل in-place computation

Figure 9.11 Flow graph of 8-point DFT using the butterfly computation of Figure 9.10.

bit reversed order

normal order

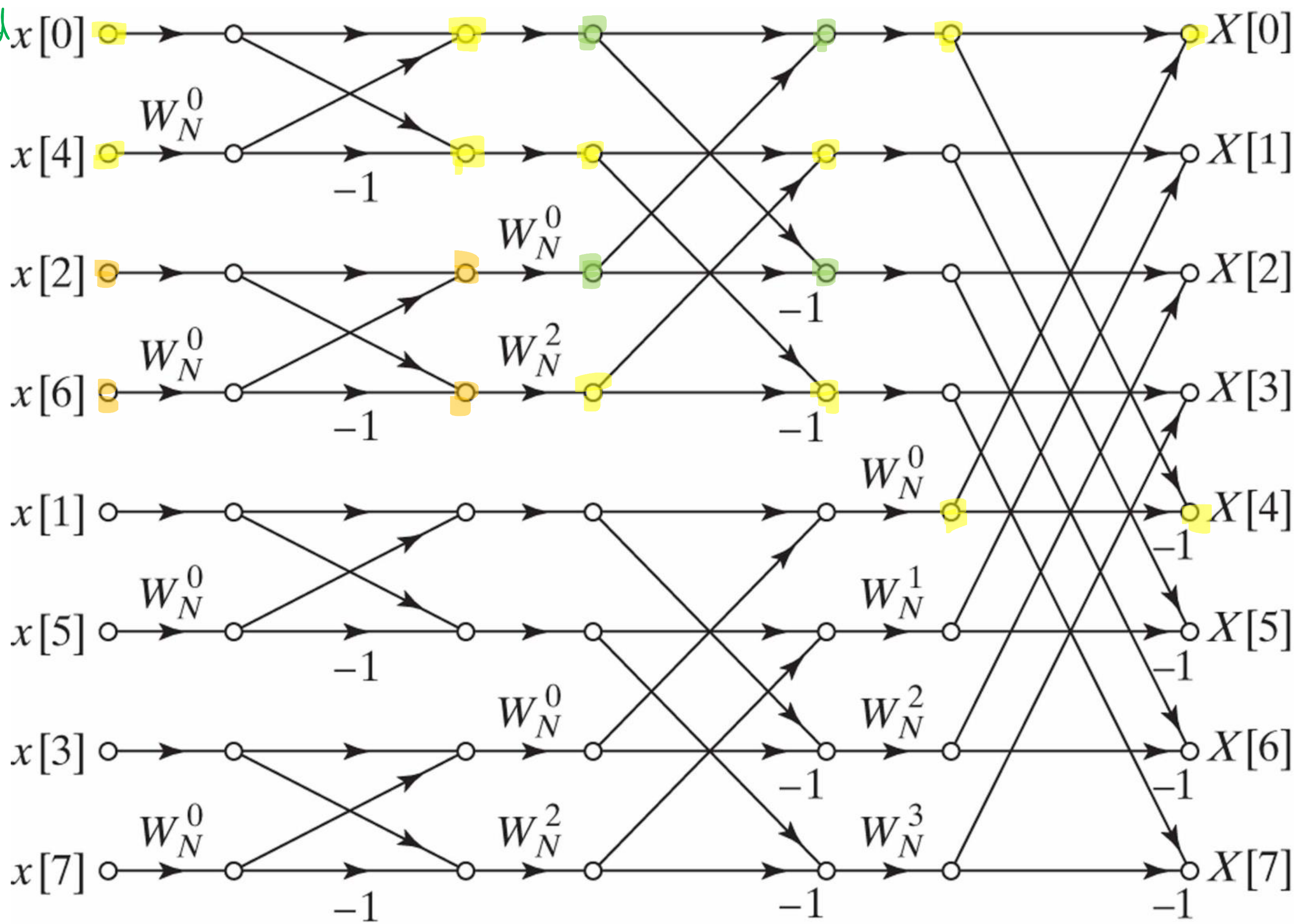


Figure 9.12 Flow graph of Eqs. (9.28).

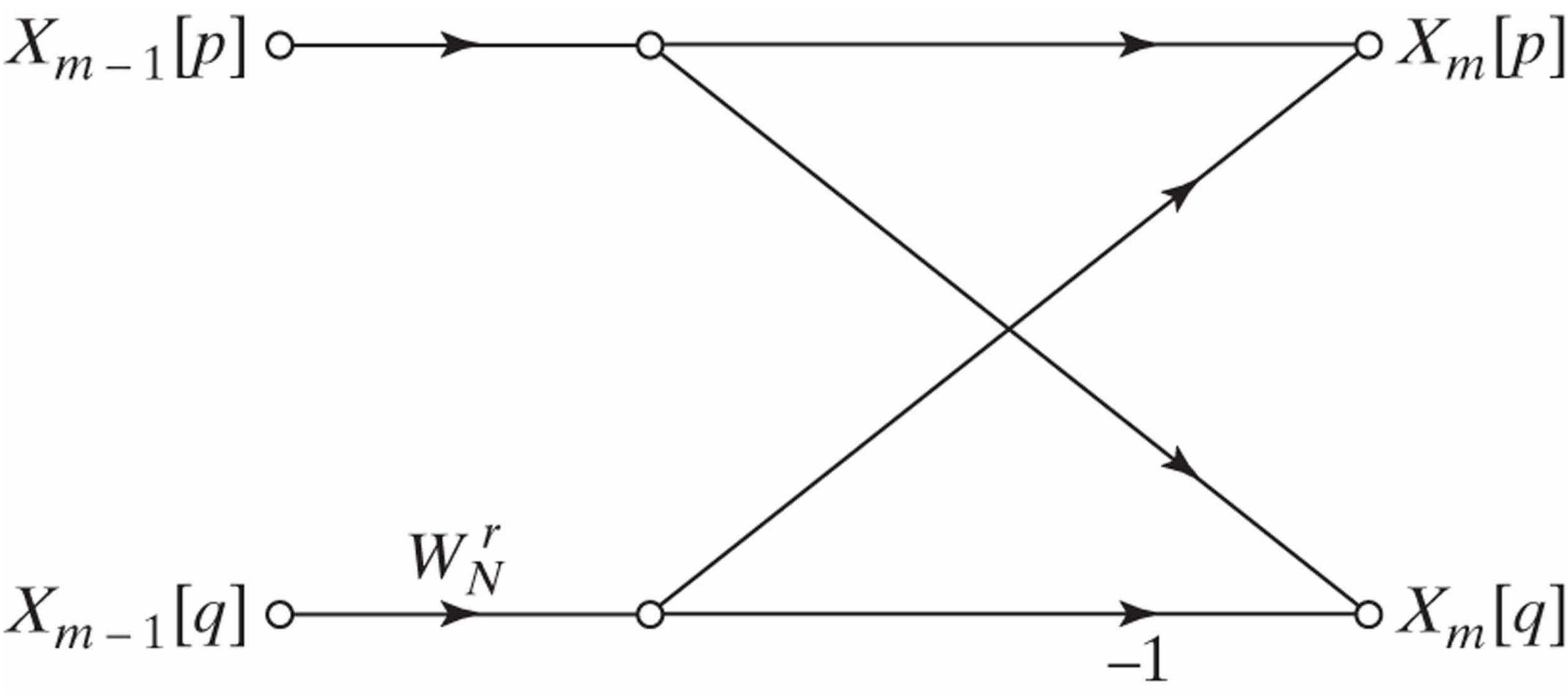


Figure 9.13 Tree diagram depicting normal-order sorting.

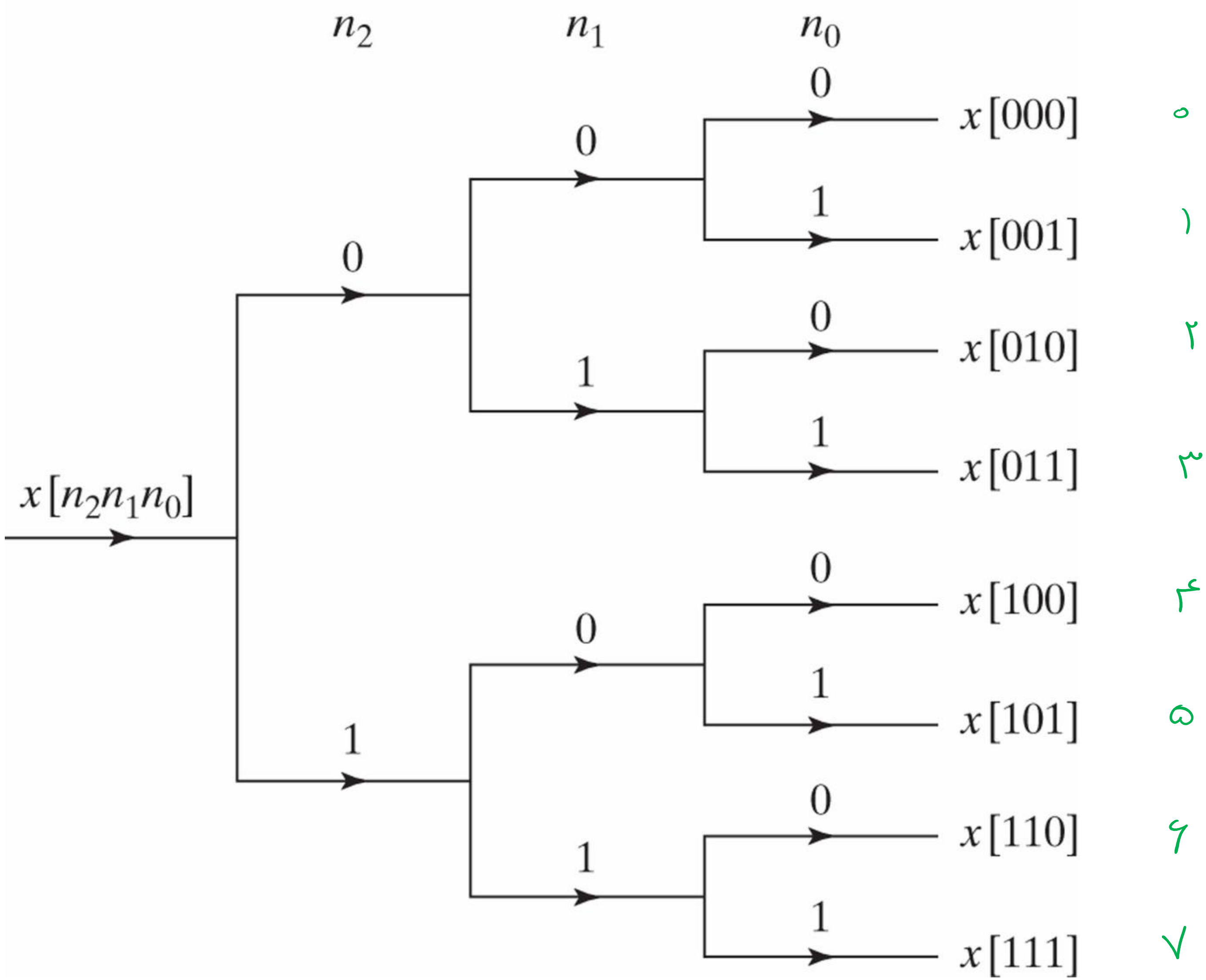


Figure 9.14 Tree diagram depicting bit-reversed sorting.

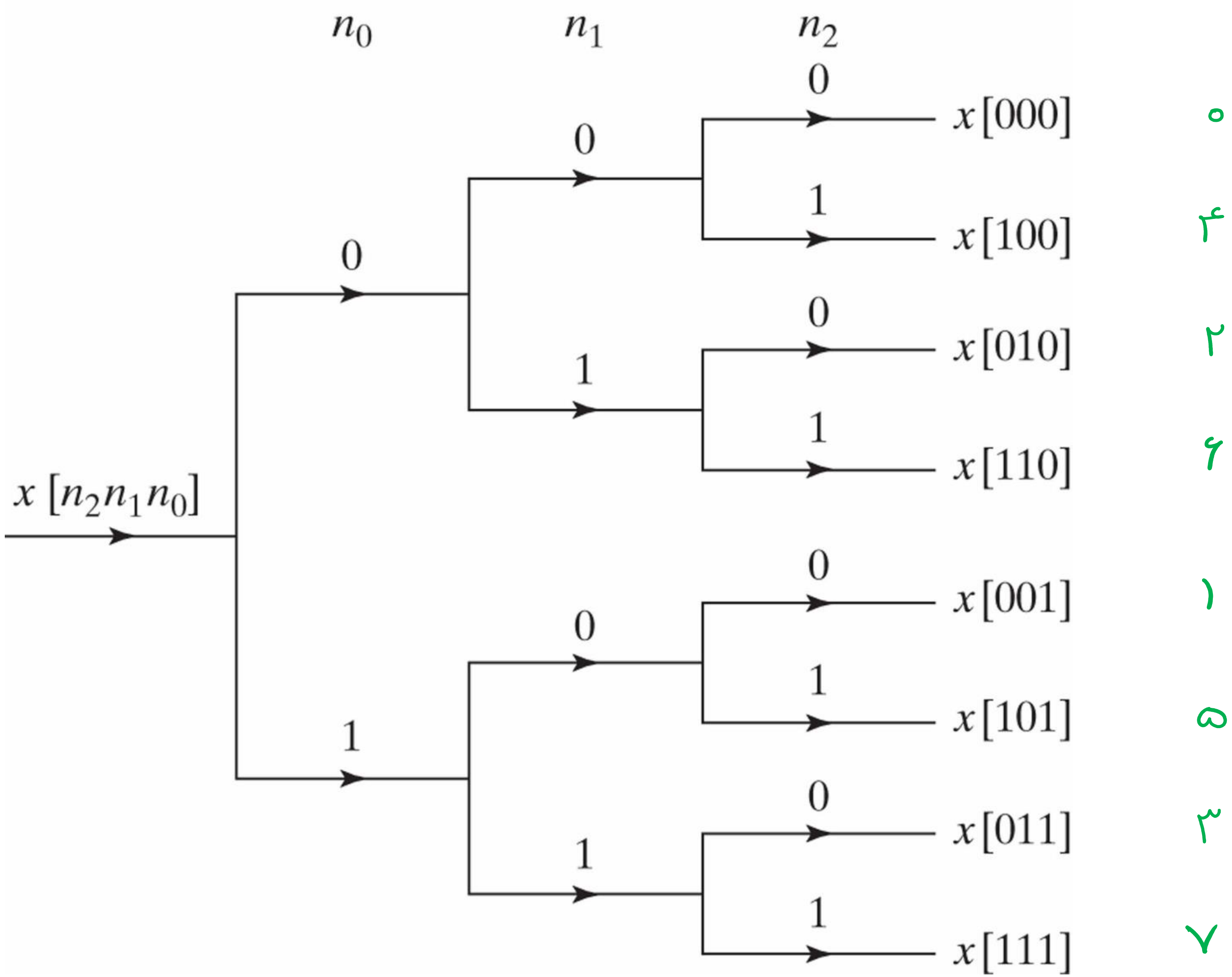


Figure 9.15 Rearrangement of Figure 9.11 with input in normal order and output in bit-reversed order.

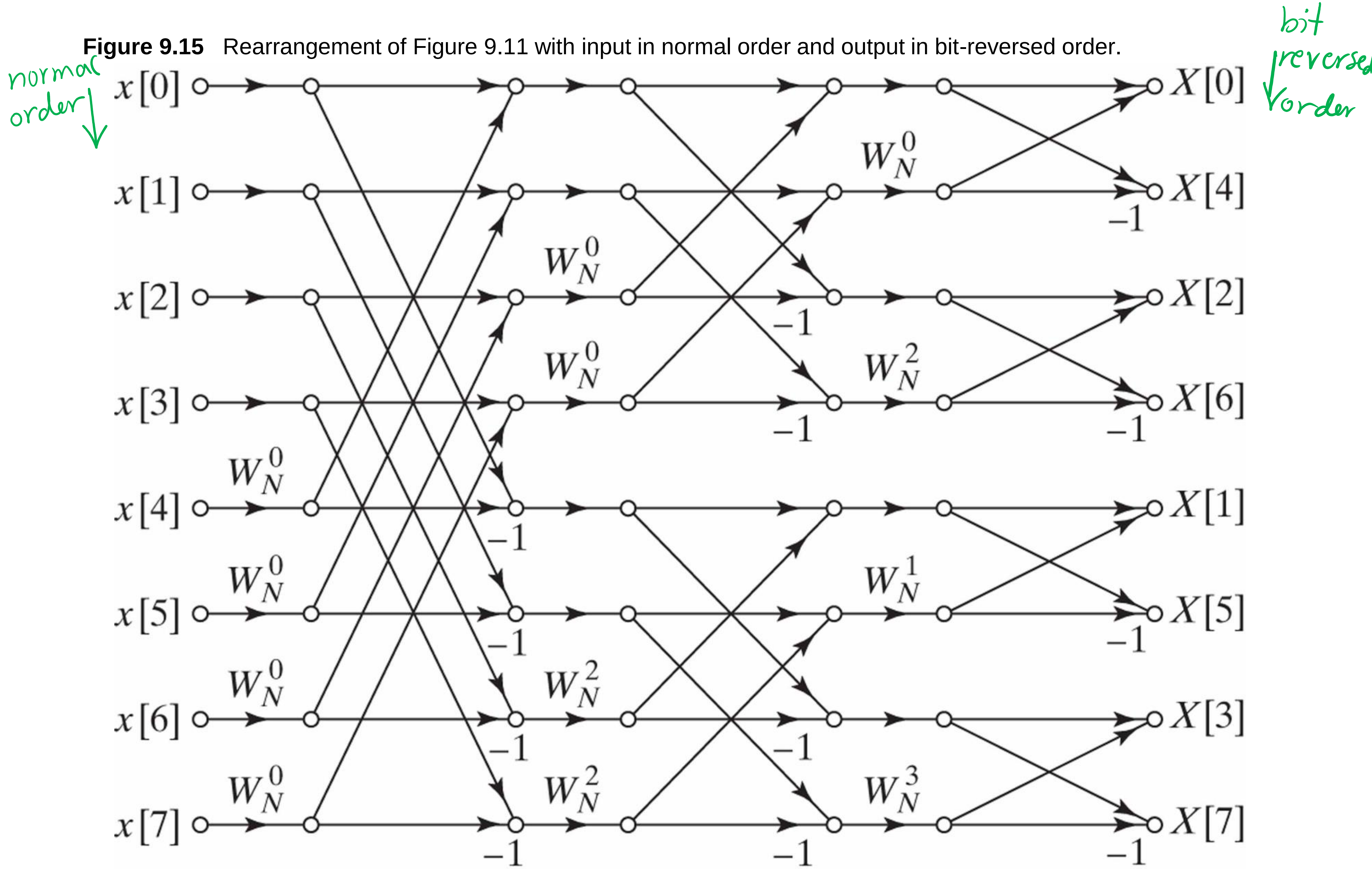


Figure 9.16 Rearrangement of Figure 9.11 with both input and output in normal order.

normal order

normal order

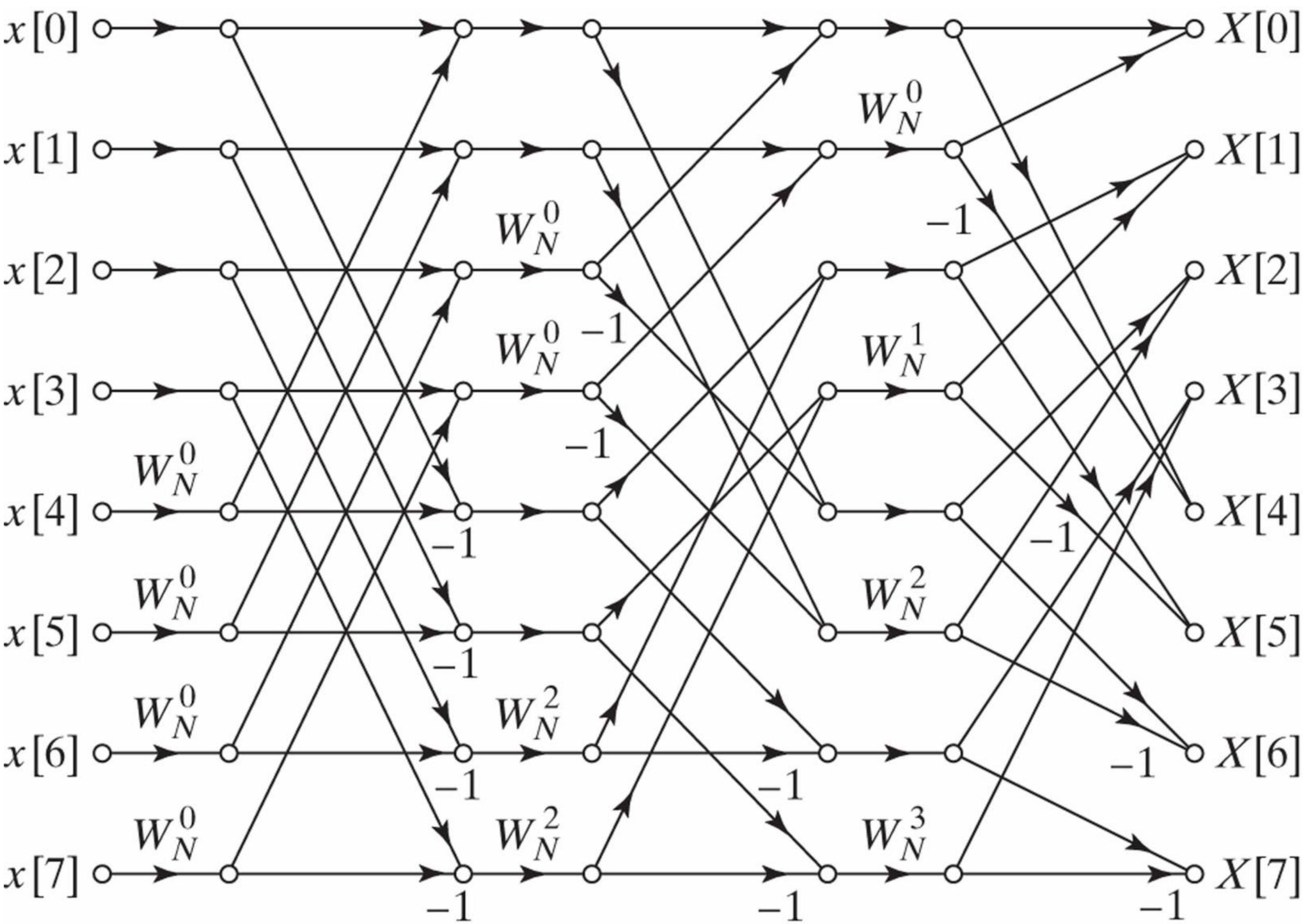
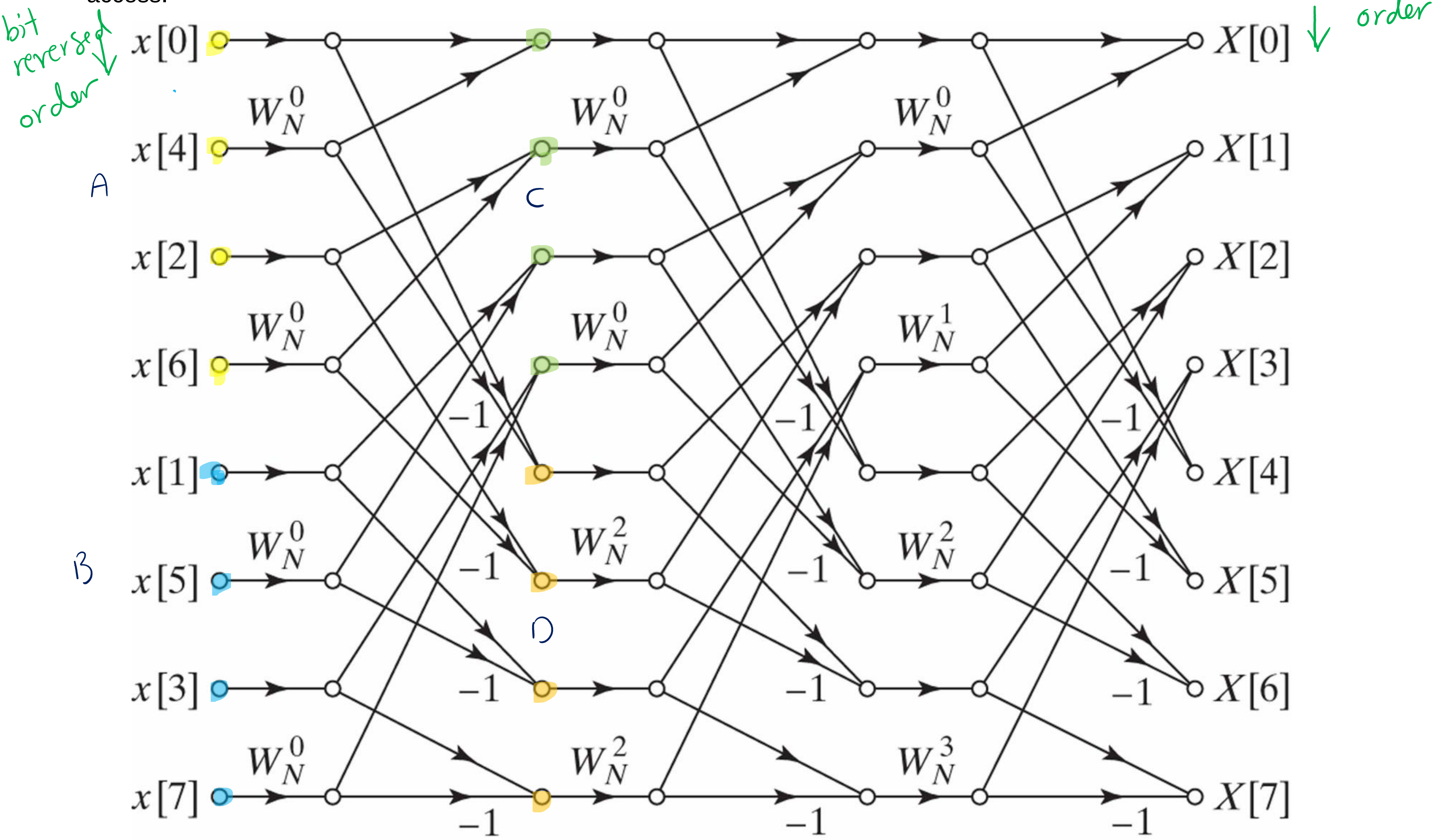


Figure 9.17 Rearrangement of Figure 9.11 having the same geometry for each stage, thereby simplifying data access.



9.3 الگوریتم FFT به روش : Decimation in Frequency

$N = 2^{\nu}$

$$X_0[k] = \sum_{n=0}^{\frac{N}{2}-1} \underbrace{\left(x[n] + x[n+\tfrac{N}{2}]\right)}_{g[n]} W_{\frac{N}{2}}^{nk}$$

$k = 0, 1, \ldots, \frac{N}{2}-1$

$$X_1[k] = \sum_{n=0}^{\frac{N}{2}-1} \left[\underbrace{\left(x[n] - x[n+\tfrac{N}{2}]\right)}_{h[n]} W_N^{n}\right] W_{\frac{N}{2}}^{nk}$$

$O(N \log_2 N)$

Figure 9.18 Illustration of the basic principle of decimation-in-frequency.

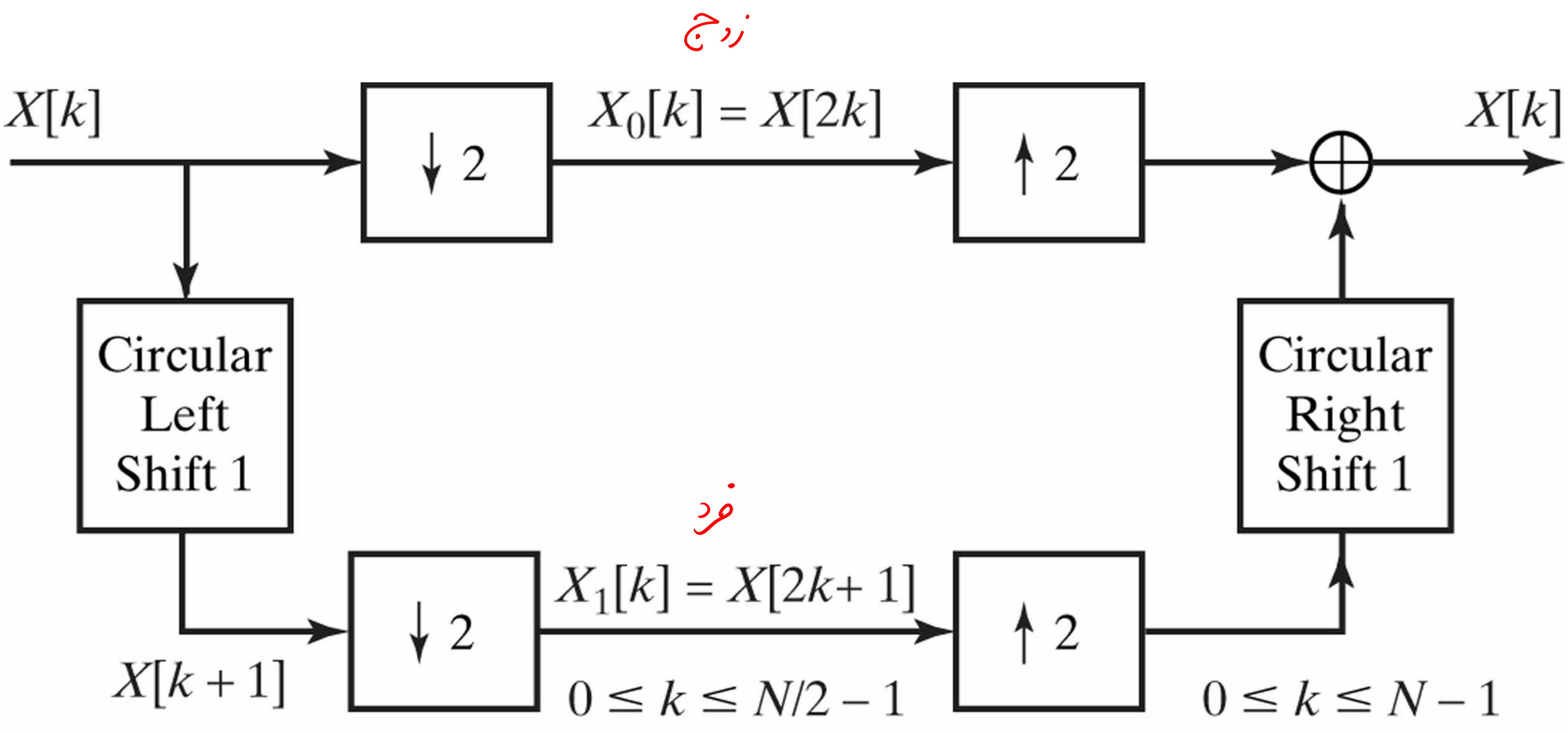


Figure 9.19 Flow graph of decimation-in-frequency decomposition of an *N*-point DFT computation into two (*N*/2)-point DFT computations ($N = 8$).

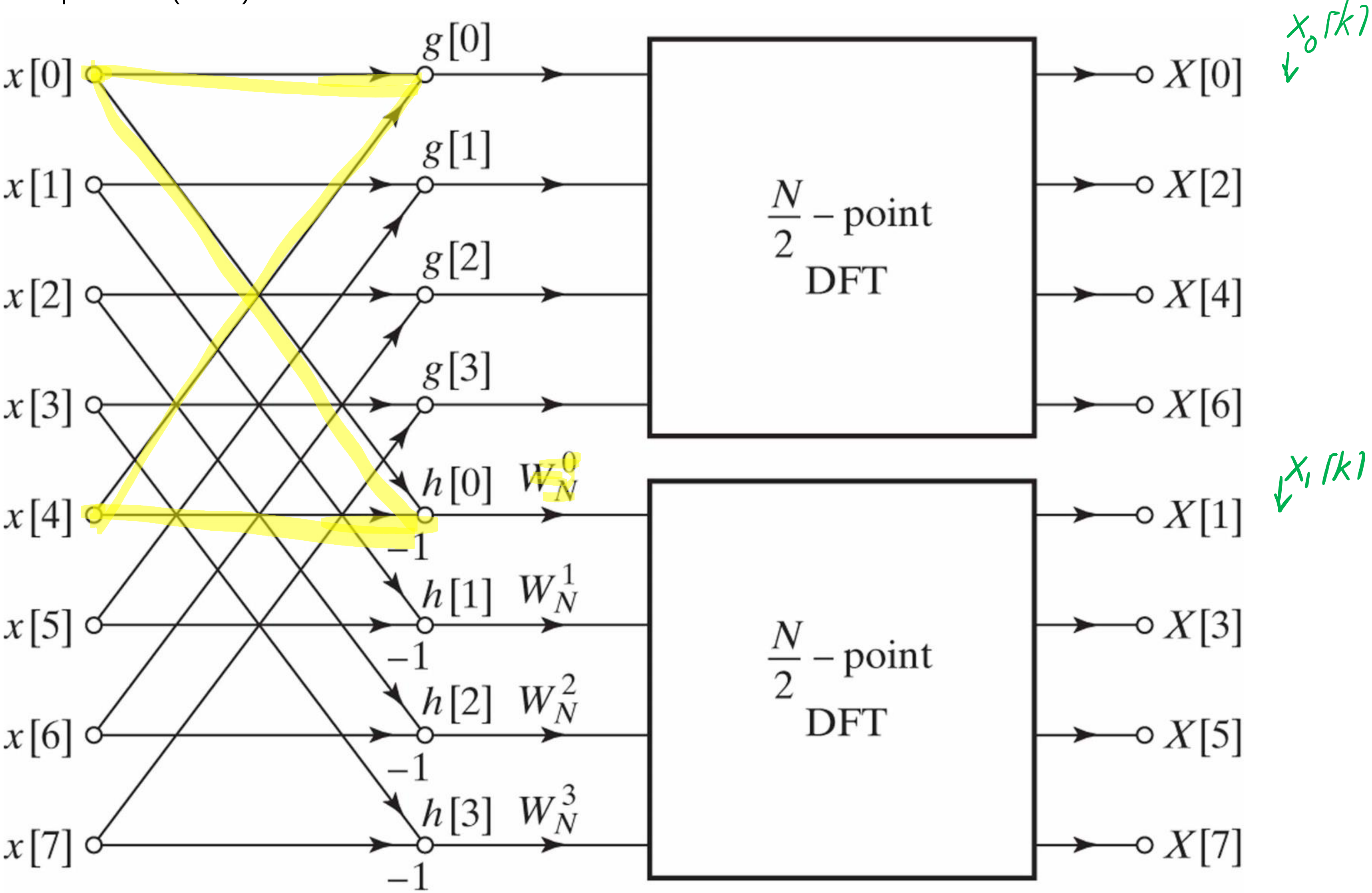


Figure 9.20 Flow graph of decimation-in-frequency decomposition of an 8-point DFT into four 2-point DFT computations.

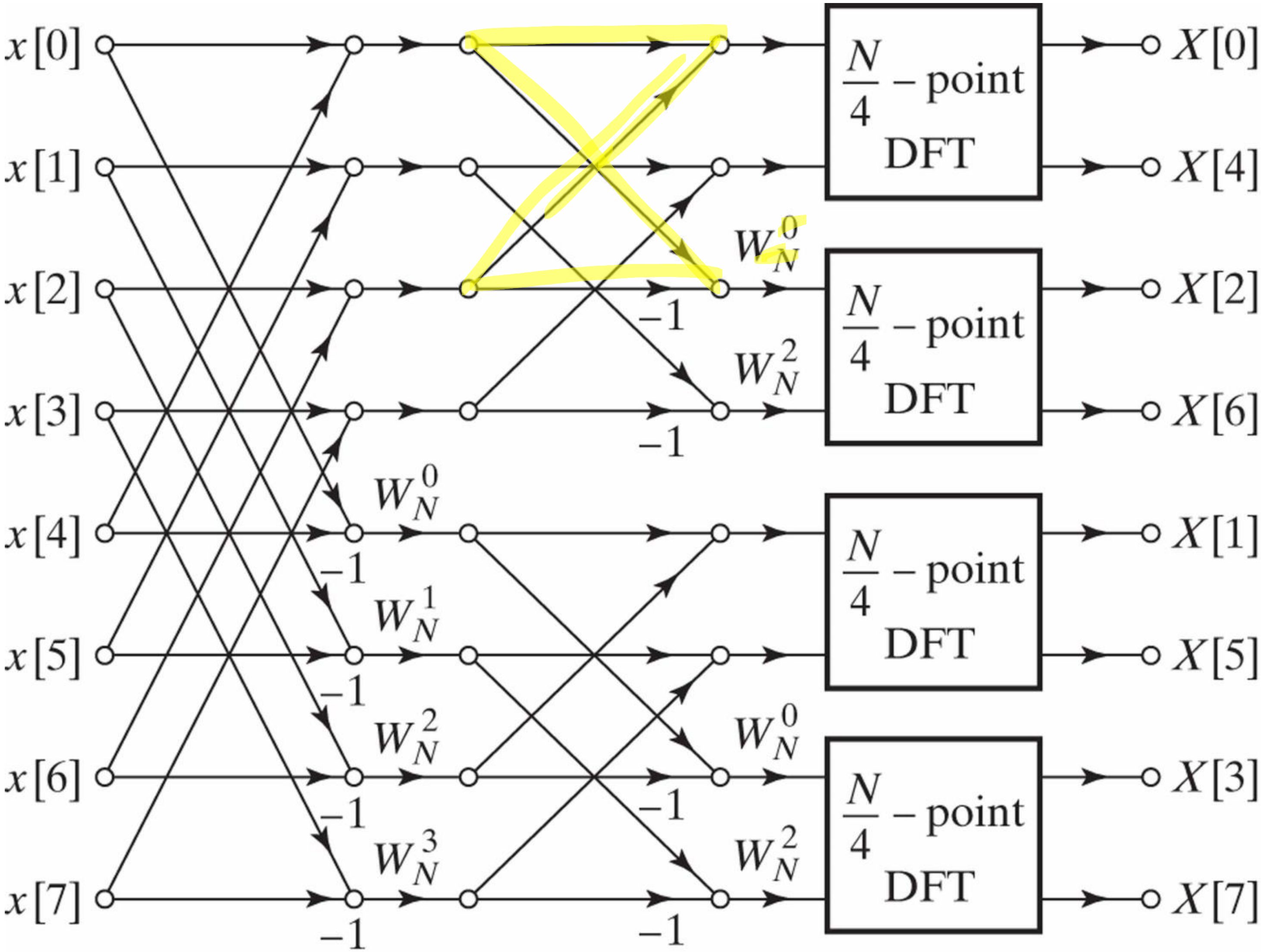


Figure 9.21 Flow graph of a typical 2-point DFT as required in the last stage of decimation-in-frequency decomposition.

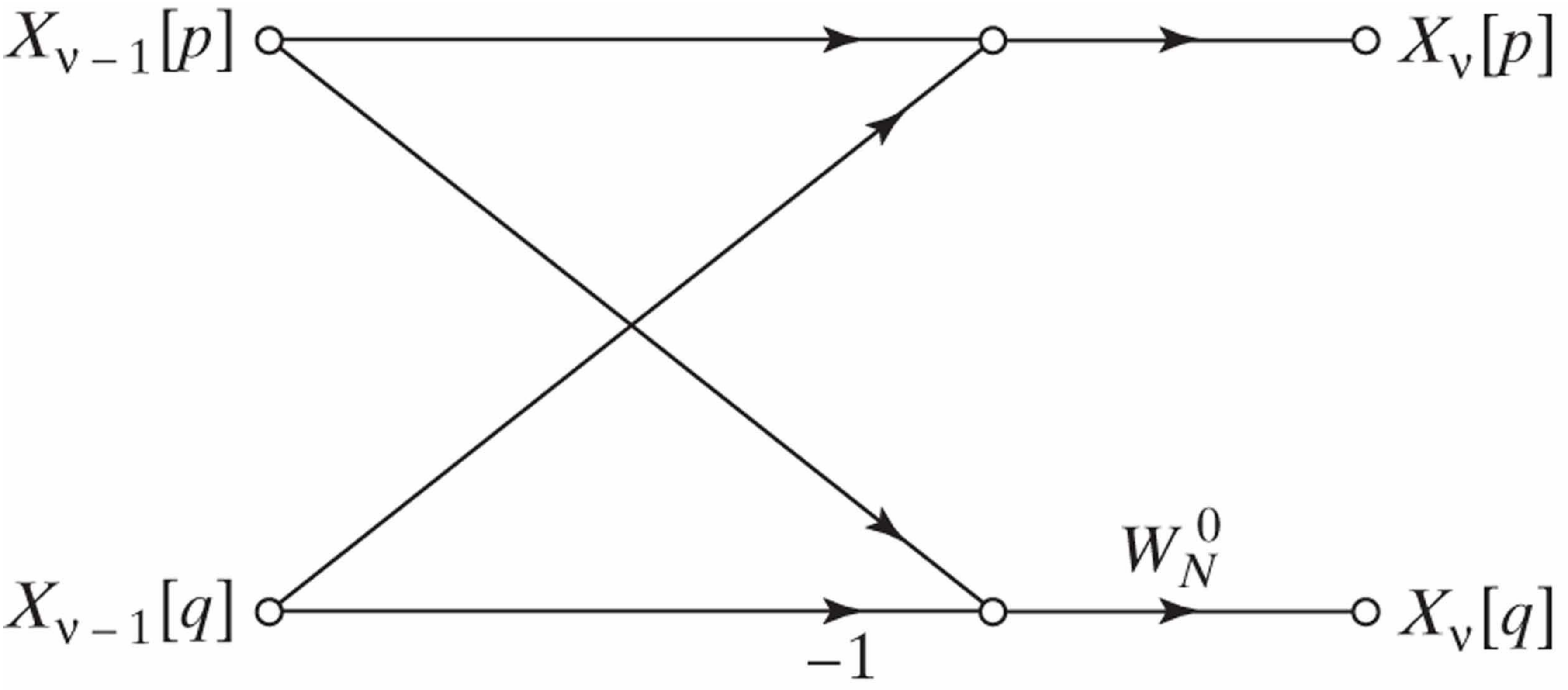


Figure 9.22 Flow graph of complete decimation-in-frequency decomposition of an 8-point DFT computation.

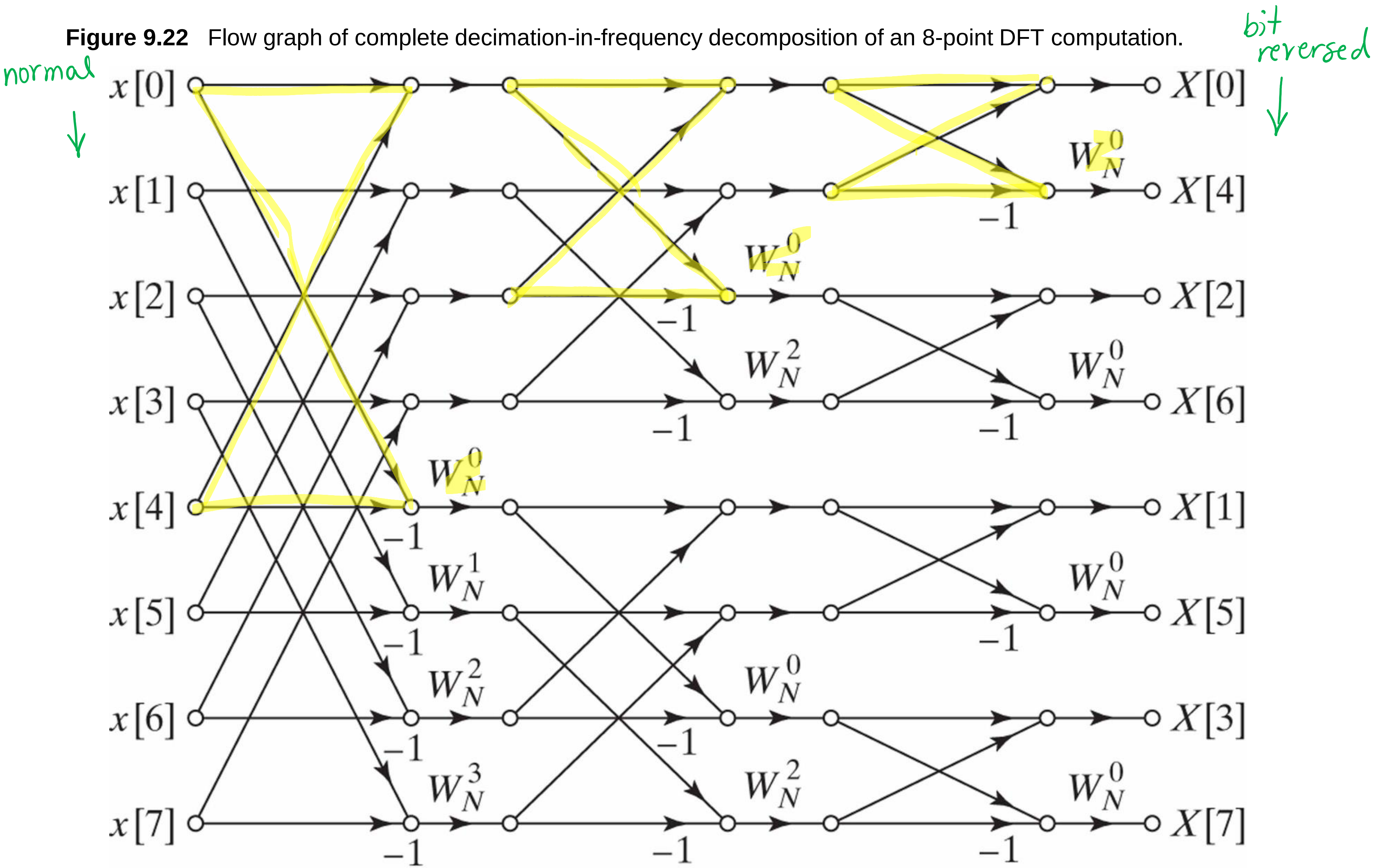


Figure 9.23 Flow graph of a typical butterfly computation required in Figure 9.22.

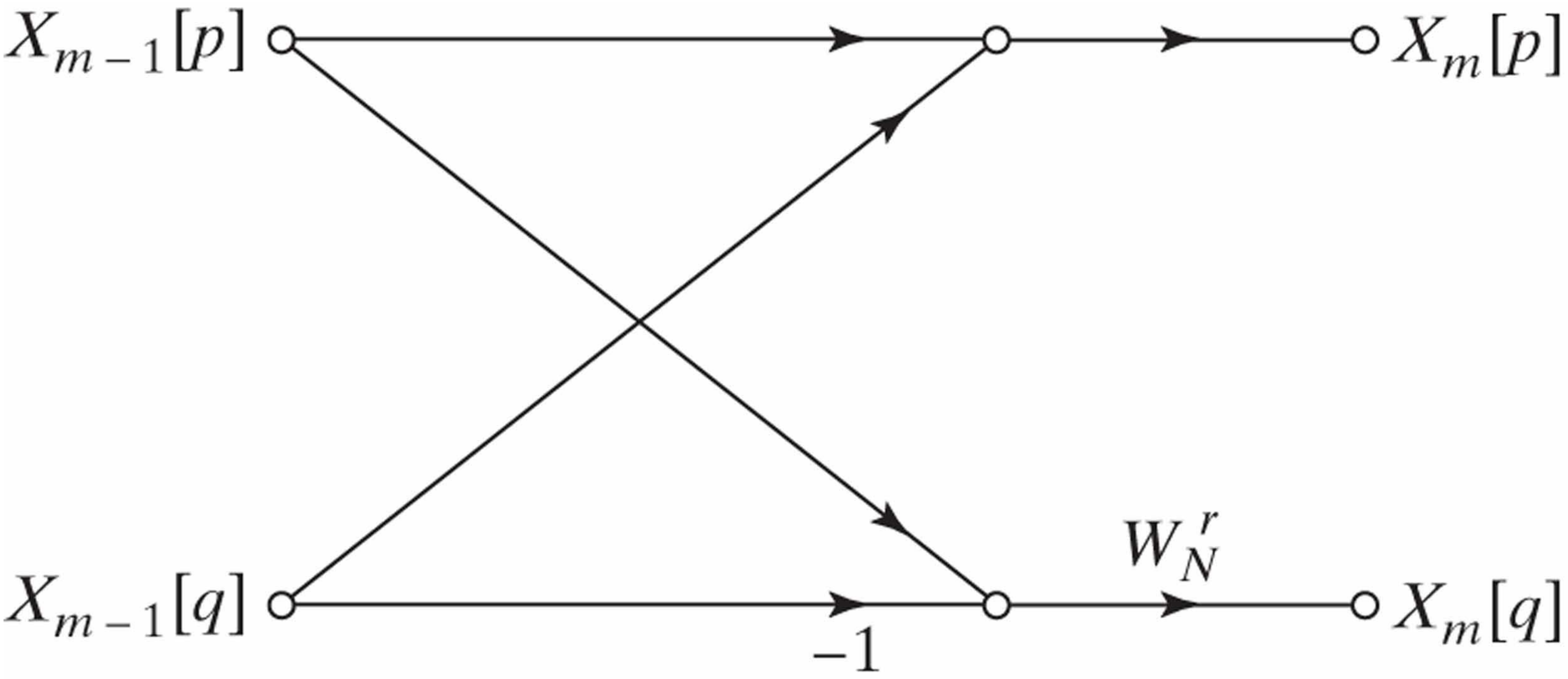


Figure 9.24 Flow graph of a decimation-in-frequency DFT algorithm obtained from Figure 9.22. Input in bit-reversed order and output in normal order. (Transpose of Figure 9.15.)

bit reversed

normal

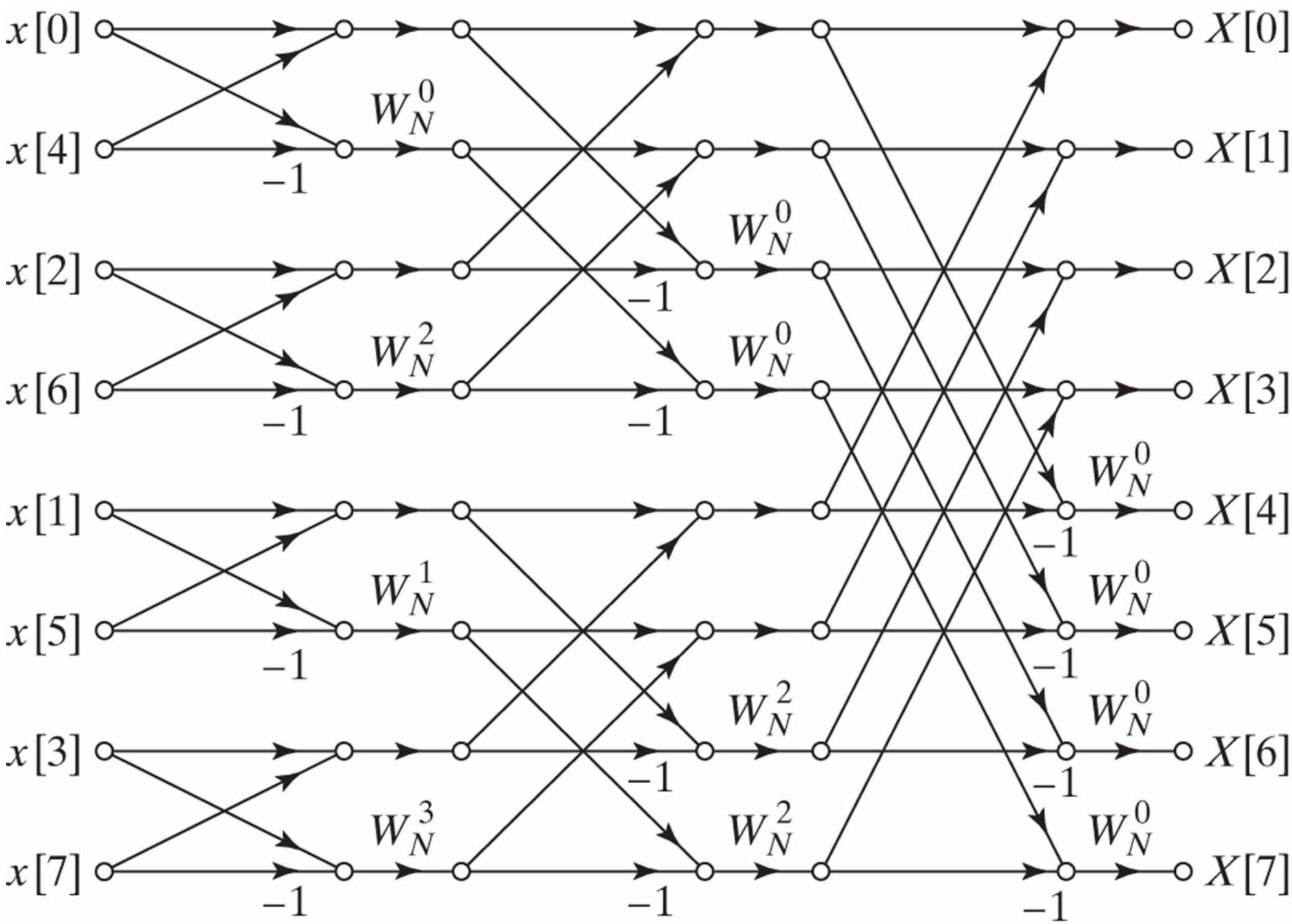


Figure 9.25 Rearrangement of Figure 9.22 having the same geometry for each stage, thereby simplifying data access. (Transpose of Figure 9.17.)

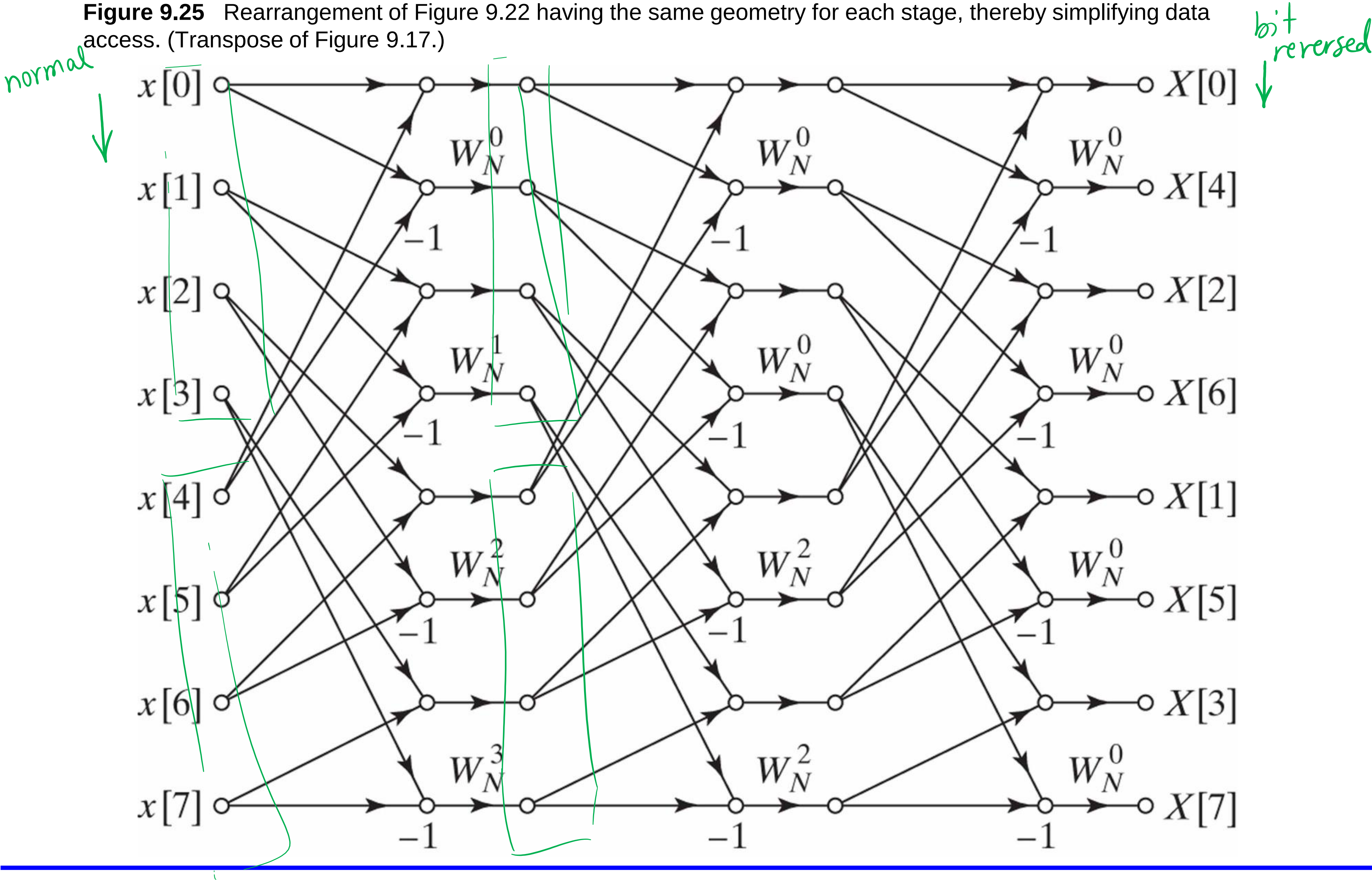
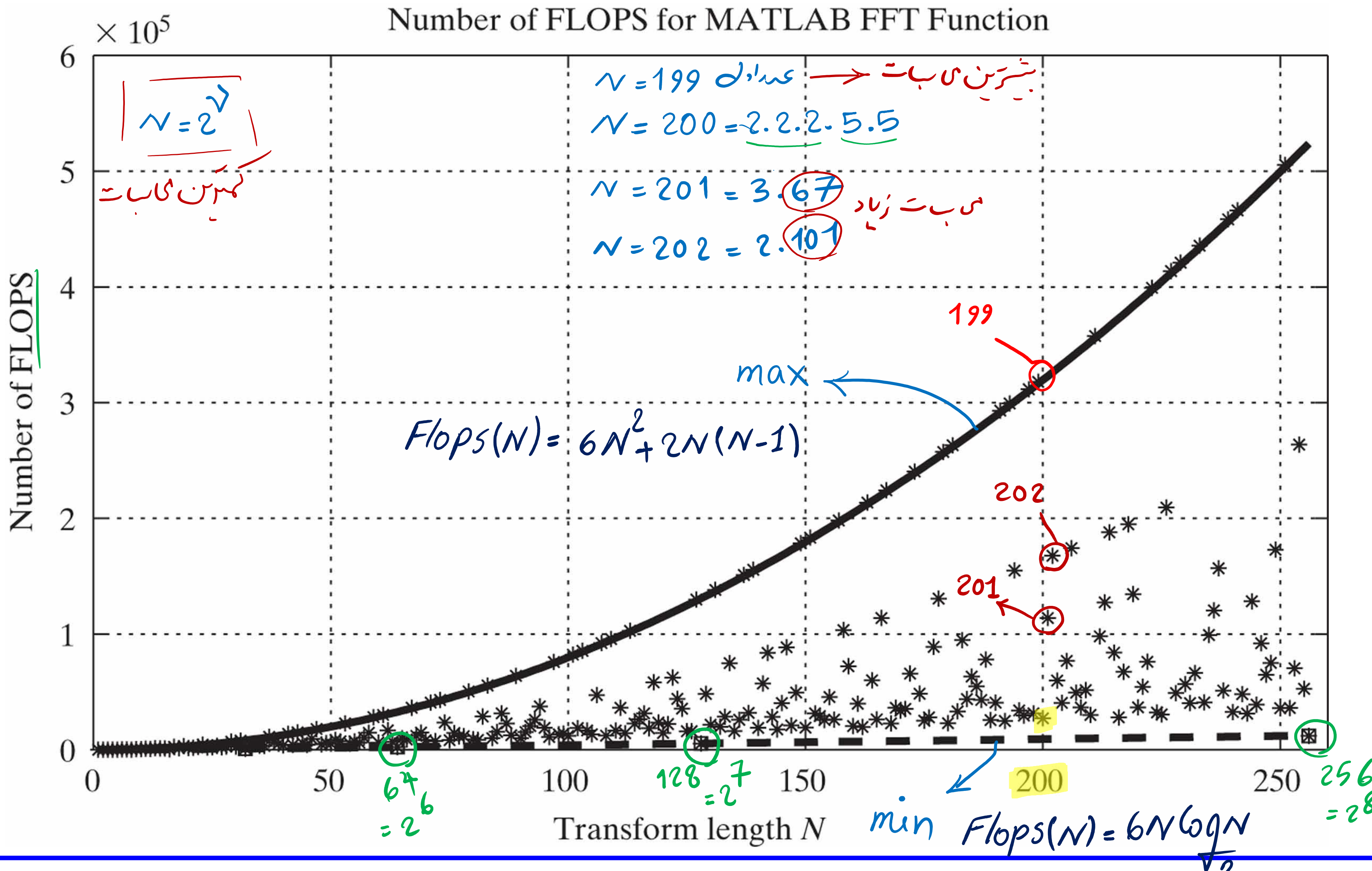


Figure 9.26 Number of floating-point operations as a function of N for MATLAB fft ( ) function (revision 5.2).



محاسبهٔ DFT با استفاده از الگوریتم‌های مبتنی بر کانولوشن :

FFT ← $N=2^{\nu}$

الگوریتم‌های FFT همهٔ ضرایب را باهم به دست می‌آورند حتی اگر تنها به تعدادی از آنها نیاز باشد.

← برای حل این مشکلات از الگوریتم‌های Convolution based استفاده می‌کنیم.

N غیر از 2^{ν}

فقط M مؤلفه

نکته: اگر $N=2^{\nu}$ و همهٔ مؤلفه‌ها را بخواهیم الگوریتم‌های FFT بهتر عمل می‌کنند.

الگوریتم Goertzel :

$W_N^{-kN} = e^{j(\frac{2\pi}{N})Nk} = e^{j2\pi k} = 1$ و ویژگی متناوب بودن

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} \xrightarrow{\times W_N^{-kN}} X[k] = W_N^{-kN} \sum_{r=0}^{N-1} x[r] W_N^{rk}$$

$$X[k] = \sum_{r=0}^{N-1} x[r] W_N^{-k(N-r)} ,$$

تعریف:

$$y_k[n] = \sum_{r=-\infty}^{+\infty} x[r] W_N^{-k(n-r)} u[n-r]$$

$$X[k] = y_k[n]\Big|_{n=N}$$

$$y_k[n] = W_N^{-k}\, y_k[n-1] + x[n] \longrightarrow \text{Fig 9.1}$$

$$X[k] = \sum_{r=0}^{N-1} x[r]\, W_N^{-k(N-r)} ,$$

تعریف:

$$y_k[n] = \sum_{r=-\infty}^{+\infty} x[r]\, \underbrace{W_N^{-k(n-r)}\, u[n-r]}_{h_k[r]}$$

$$X[k] = y_k[n]\big|_{n=N}$$

$$\Rightarrow y_k[n] = x[n] * h_k[n]$$

$$X[k] = y_k[n]\big|_{n=N}$$

Figure 9.1 Flow graph of 1st-order complex recursive computation of $X[k]$.

$$y_k[n] = W_N^{-k} y_k[n-1] + x[k] \quad , \quad H_k(z) = \frac{1 \times (1 - W_N^{-k} z^{-1})}{1 - W_N^{-k} z^{-1} \times (1 - W_N^{-k} z^{-1})}$$

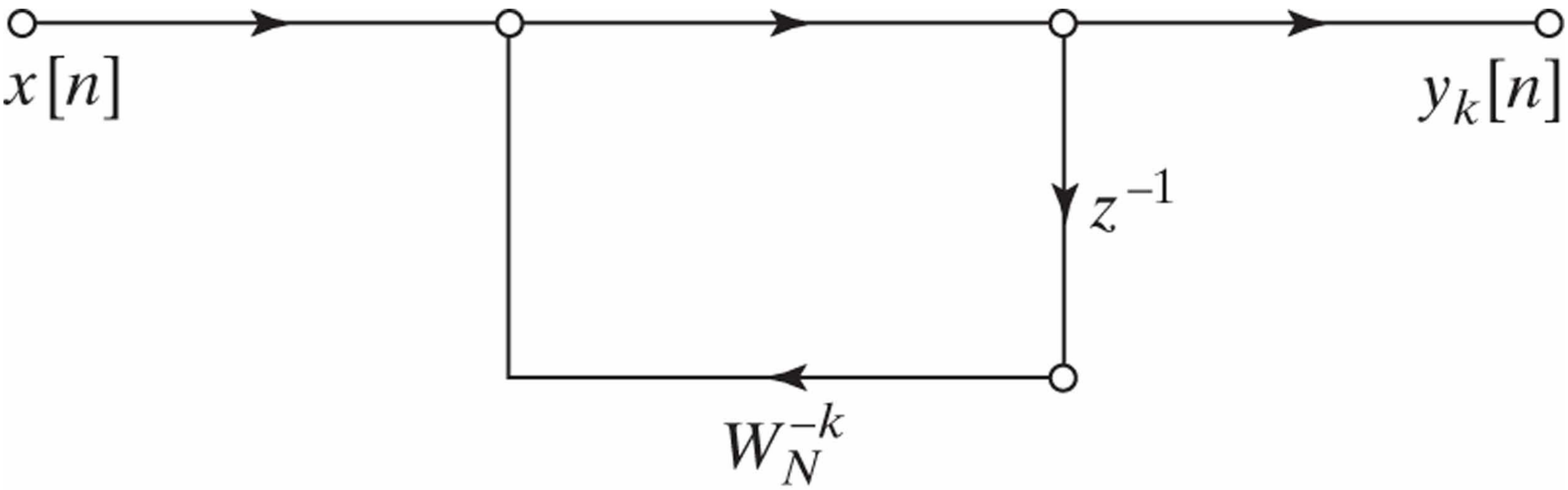


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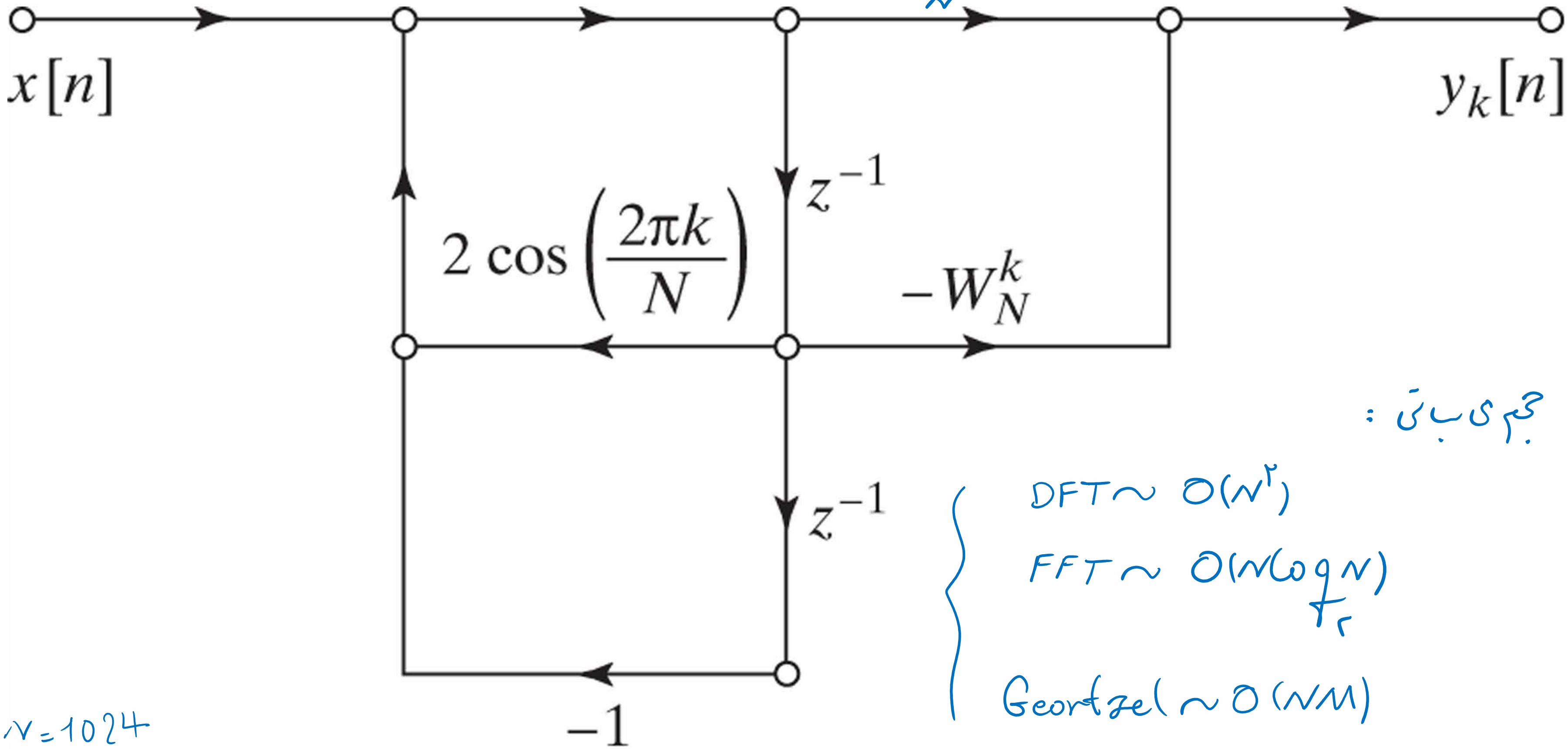
Figure 9.2 Flow graph of 2nd-order recursive computation of $X[k]$ (Goertzel algorithm).

$$H(z) = \frac{1 - W_N^k z^{-1}}{1 - 2\cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}}$$

پیاده سازی SFG
از فصل ۶ :



حجم محاسباتی :

$DFT \sim O(N^2)$

$FFT \sim O(N \log_2 N)$

$Geortzel \sim O(NM)$

$N = 1024$

$M < 10$

در چه حالتی از FFT بهتر است ؟

اگر $M < \log_2 N$ باشد از FFT بهتر است

الگوریتم chirp :

قضیه: با استفاده از سیستم شکل 9.33 می‌توان $X[k]$ را به صورت زیر به دست آورد:

$$x[n] \; ; \quad 0 \le n \le N-1$$

$$y_k[n]\Big|_{n=k+N} = X[k] \qquad k = 0, 1, \ldots, N-1$$

$$h_2[n] = \begin{cases} W^{-n^2/2} & ; \; -(N-1) \le n \le (M-1) \\ 0 & ; \; \text{o.w.} \end{cases}$$

$$0 \le M \le N$$

اثبات :

الگوریتم تبدیل chirp : CTA

chirp Transform Algorithm

دنباله $x[n]$ را با N نقطه فرض می‌کنیم، و تبدیل فوریه آن DTFT $X(e^{j\omega})$

هدف : محاسبه M نمونه از $X(e^{j\omega})$ است که با گام‌های ثابت روی دایره واحد قرار گرفته‌اند (Fig 9.27)

$$\omega_k = \omega_0 + k\Delta\omega \quad ; \quad k = 0, 1, \dots, M-1$$

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n} \quad ; \quad k = 0, 1, \dots, M-1$$

$$\omega_k = \omega_0 + k\Delta\omega \longrightarrow X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega_0 n}\, e^{-jk\Delta\omega n}$$

$$\boxed{e^{-j\Delta\omega} = W} \longrightarrow X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega_0 n}\, W^{nk}$$

$$nk = \frac{1}{2}[n^2 + k^2 - (k-n)^2] \longrightarrow X(e^{j\omega_k}) = \sum_{n=0}^{N-1} \underbrace{x[n] e^{-j\omega_0 n}\, W^{n^2/2}}_{g[n]}\, W^{k^2/2}\, W^{-\frac{(k-n)^2}{2}}$$

$$X(e^{j\omega_k}) = W^{k^2/2}\left(\sum_{n=0}^{N-1} g[n]\, W^{-\frac{(k-n)^2}{2}}\right) \; ; \; k = 0, 1, \ldots, M-1$$

$$n \leftrightarrow k \longrightarrow \quad X(e^{j\omega_n}) = \underbrace{W^{n^2/2}}_{\text{ضرب}}\underbrace{\left(\sum_{k=0}^{N-1} g[k]\, W^{-\frac{(n-k)^2}{2}}\right)}_{g[n] * W^{-\frac{n^2}{2}}} \; ; \; n = 0, 1, \ldots, M-1$$

$\rightarrow$ Fig 9.28

$$W^{-\frac{n^2}{2}} = \left(e^{-j\Delta\omega}\right)^{-\frac{n^2}{2}} = e^{\frac{1}{2}j(n\Delta\omega)n}$$

نهایی متغیر با فرکانس $n\Delta\omega$

افزایش فرکانس به صورت خطی $\rightarrow$ chirp — radar, sonar

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Figure 9.27 Frequency samples for chirp transform algorithm.

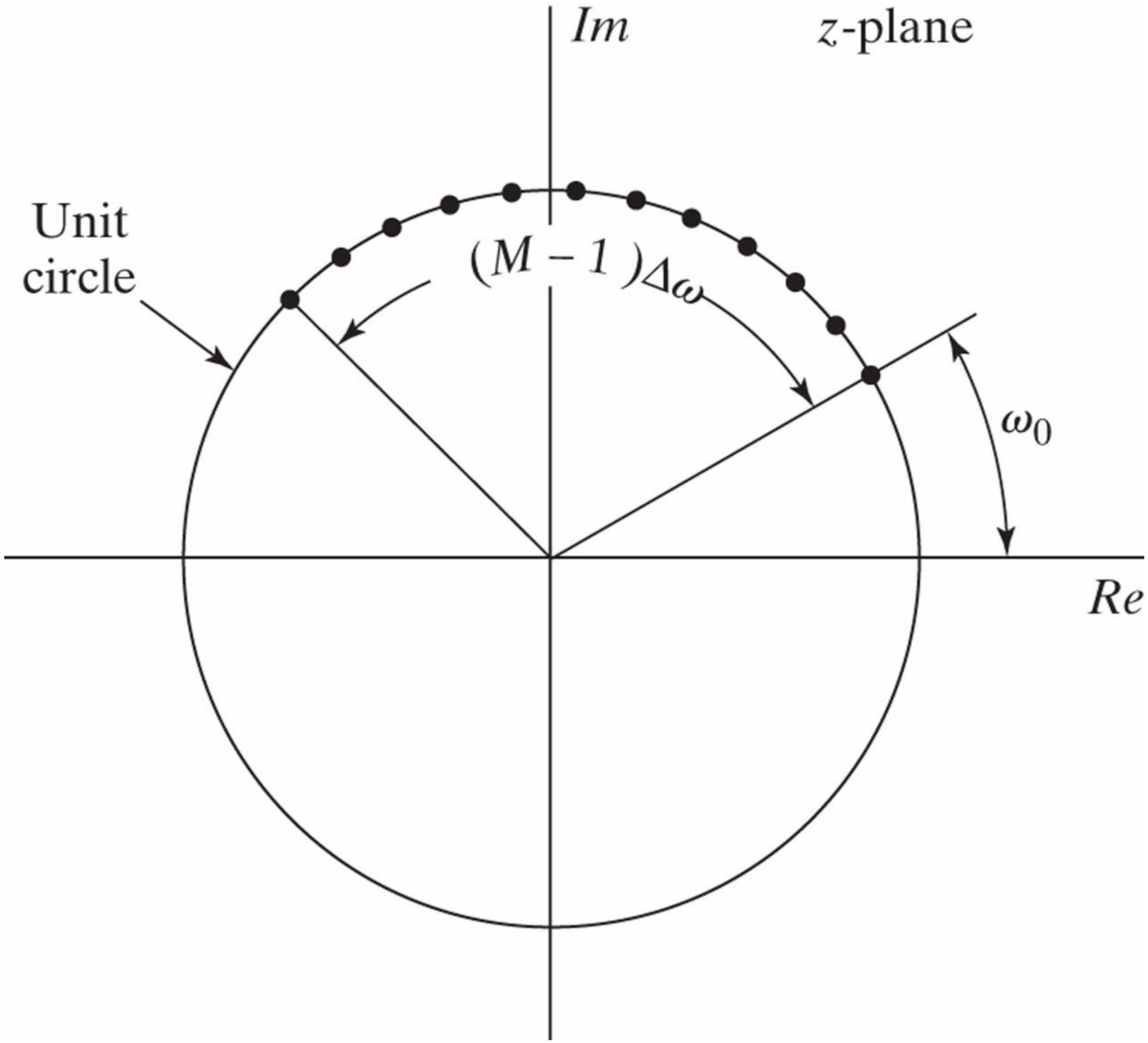


Figure 9.28 Block diagram of chirp transform algorithm.

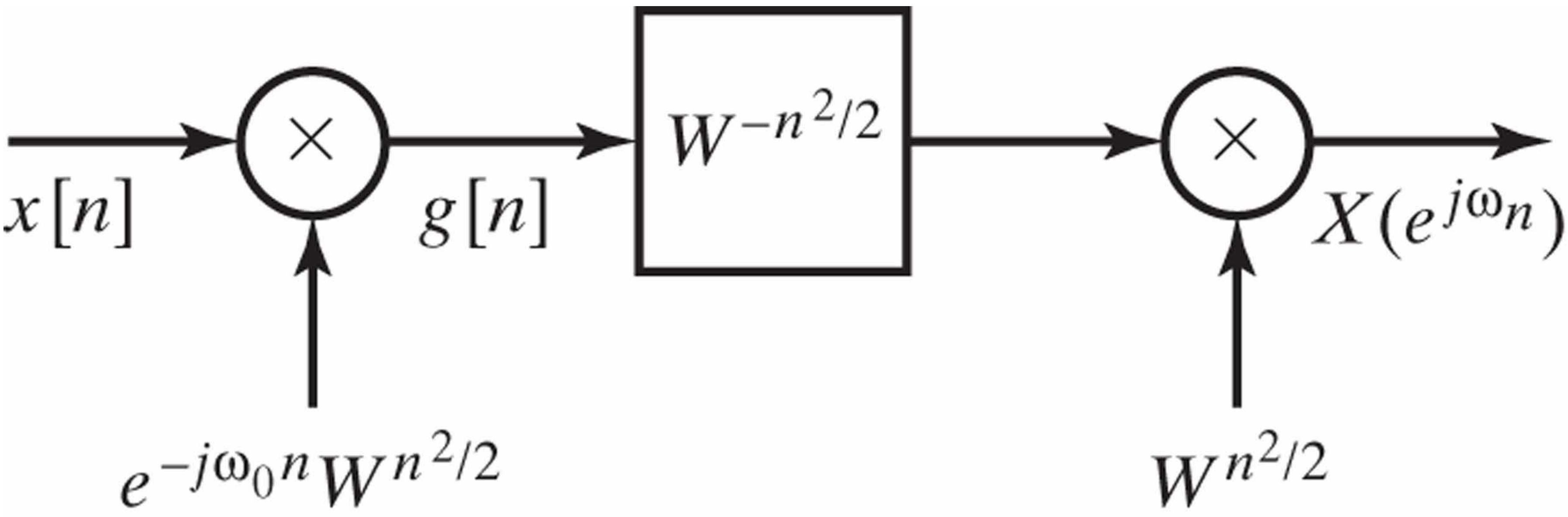


Figure 9.29 An illustration of the sequences used in the chirp transform algorithm. Note that the actual sequences involved are complex valued. (a) $g[n] = x[n]e^{-j\omega_0 n} W^{n^2/2}$. (b) $W^{-n^2/2}$. (c) $g[n]$ $W^{-n^2/2}$.

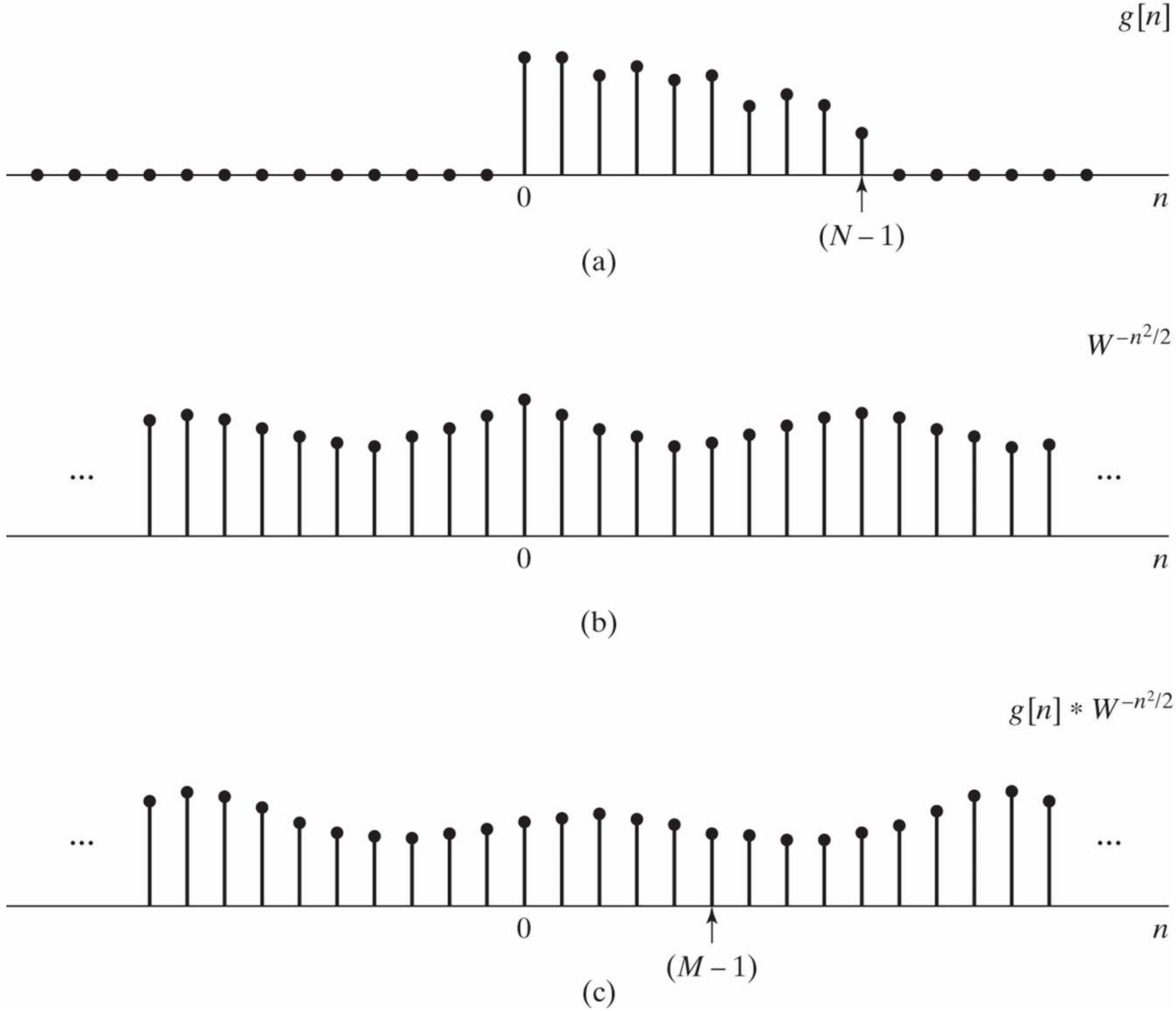


Figure 9.30 An illustration of the region of support for the FIR chirp filter. Note that the actual values of $h[n]$ as given by Eq. (9.48) are complex.

دنباله غیرعلی

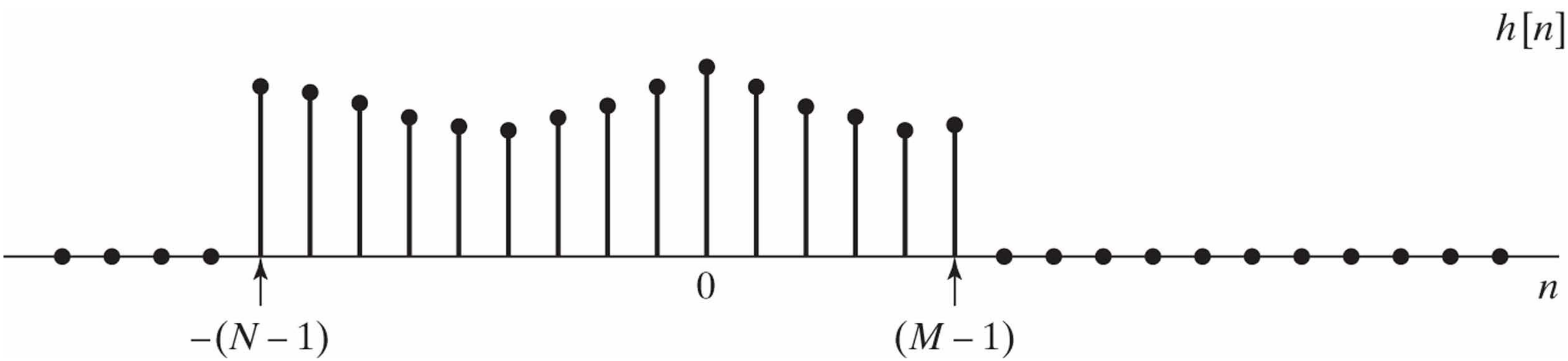


Figure 9.31 Block diagram of chirp transform system for finite-length impulse response.

$$h[n] = \begin{cases} W^{-n^2/2} ; & -(N-1) \leq n \leq (M-1) \\ 0 ; & \text{o.w.} \end{cases}$$

$$X(e^{j\omega_n}) = y[n]$$

$n = 0, 1, \ldots, M-1$

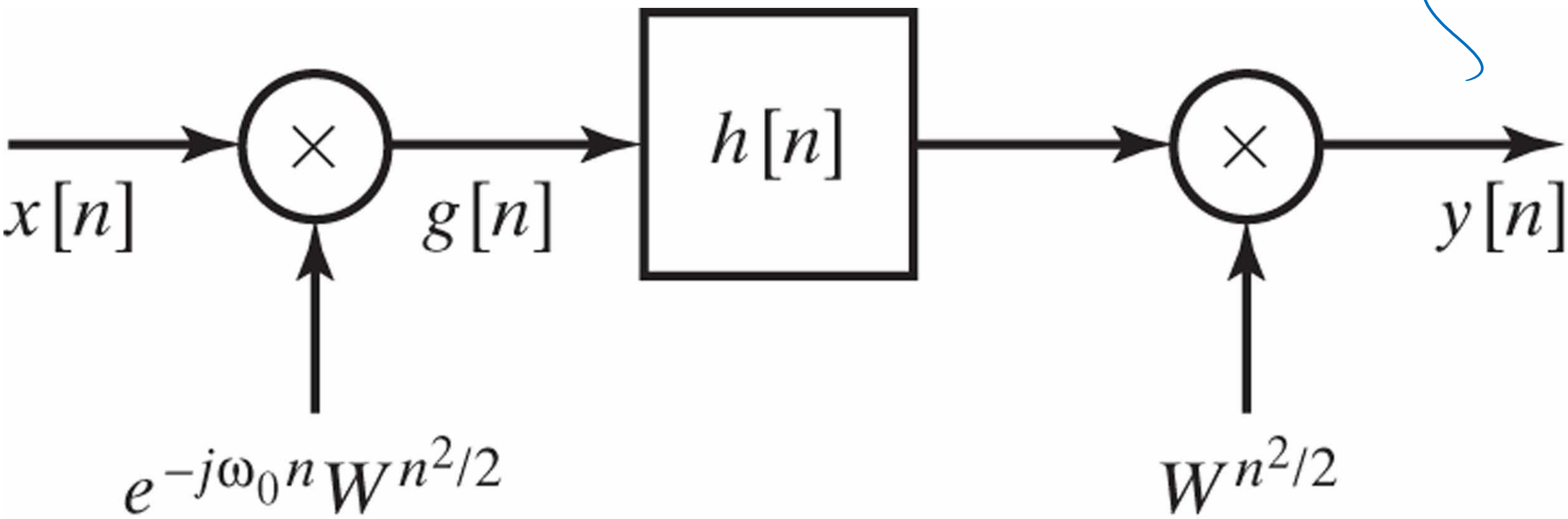


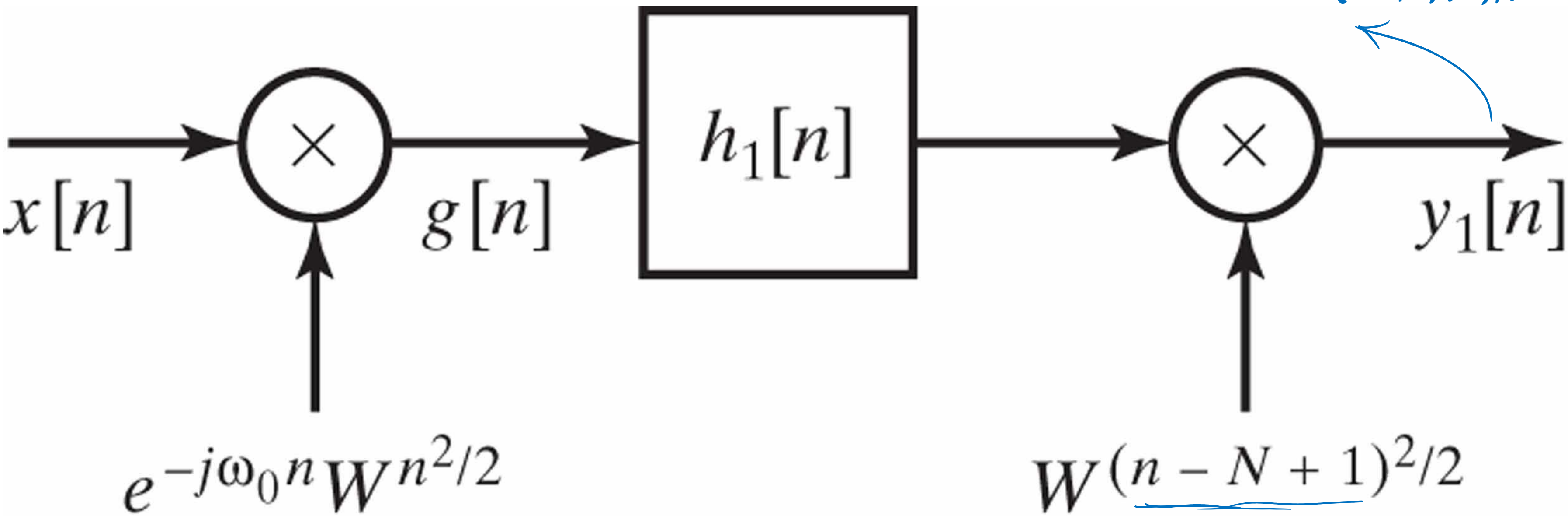
Figure 9.32 Block diagram of chirp transform system for causal finite-length impulse response.

پاسخ ضربه علّی:

$$h_1[n] = \begin{cases} W^{-\frac{(n-N+1)^2}{2}} & ; \quad n = 0,1,2,\ldots,M+N-2 \\ 0 & ; \quad \text{o.w.} \end{cases}$$

$$\rightarrow X(e^{j\omega_n}) = y_1[n+N-1]$$

$n = 0,1,\ldots,M-1$



محاسبه‌ی DFT با استفاده از الگوریتم تبدیل chirp :

Figure 9.33 Block diagram of chirp transform system for obtaining DFT samples.

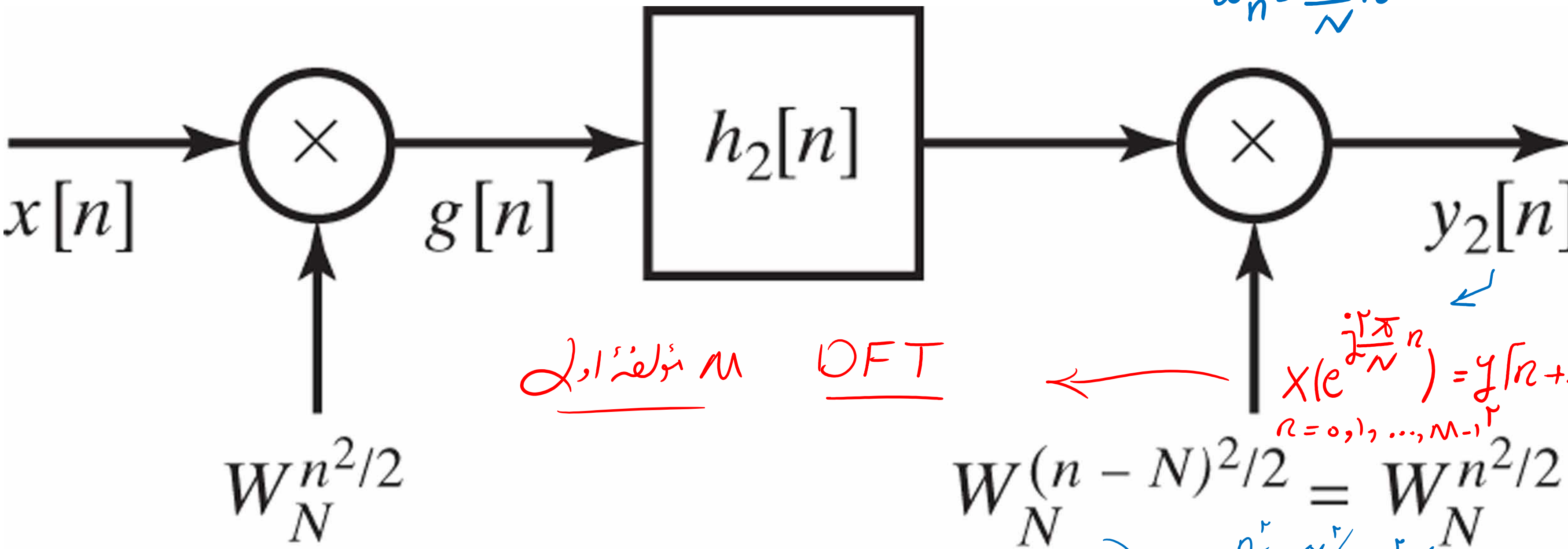
$$h_2[n] = \begin{cases} W_N^{-n^2/2} & ; \; n = 1, 2, \ldots, M+N-1 \\ 0 & ; \; o.w. \end{cases}$$

ساده‌سازی:

$\omega_0 = 0 \qquad \Delta\omega$

$W = e^{-j\frac{2\pi}{N}} = W_N$

$\omega_n = \frac{2\pi}{N} n$



$X(e^{j\frac{2\pi}{N}n}) = y[n+N]$

$n = 0, 1, \ldots, M-1$

DFT

M نقطه‌ی اول

$W_N^{n^2/2} \; \cancel{W_N^{N^2/2}} \; \cancel{W_N^{-2nN/2}}$

$1 \qquad 1$

$W_N^N = 1$

مثال: پارامترهای تبدیل chirp

فرض کنید $x[n]$ در بازه $\underbrace{n = 0, \ldots, 25}_{N=26}$ غیرصفر باشد

می‌خواهیم ۱۶ نمونه از $X(e^{j\omega})$ DTFT را در بازه $\omega_k = \frac{2\pi}{27} + \frac{2\pi k}{1024}$ به دست آوریم.

$$\omega_0 = \frac{2\pi}{27}, \quad \Delta\omega = \frac{2\pi}{1024} \qquad W = e^{-j\Delta\omega} = e^{-j\frac{2\pi}{1024}}$$

Fig 9.32 $\longrightarrow$

$$h_1[n] = \begin{cases} \left[e^{-j\frac{2\pi}{1024}}\right]^{\frac{-(n-25)^2}{2}}, & n = 0, \ldots, \overset{\overset{M+N-2}{16+26-2}}{40} \\ 0 & ;\ \text{o.w} \end{cases}$$

$$y_1[n+25] = X(e^{j\omega_n})\Big|_{\omega_n = \frac{2\pi}{27} + \frac{2\pi n}{1024}} \quad ;\ n = 0, 1, \ldots, 15$$

Figure 9.34 Flow graph for decimation-in-time FFT algorithm.

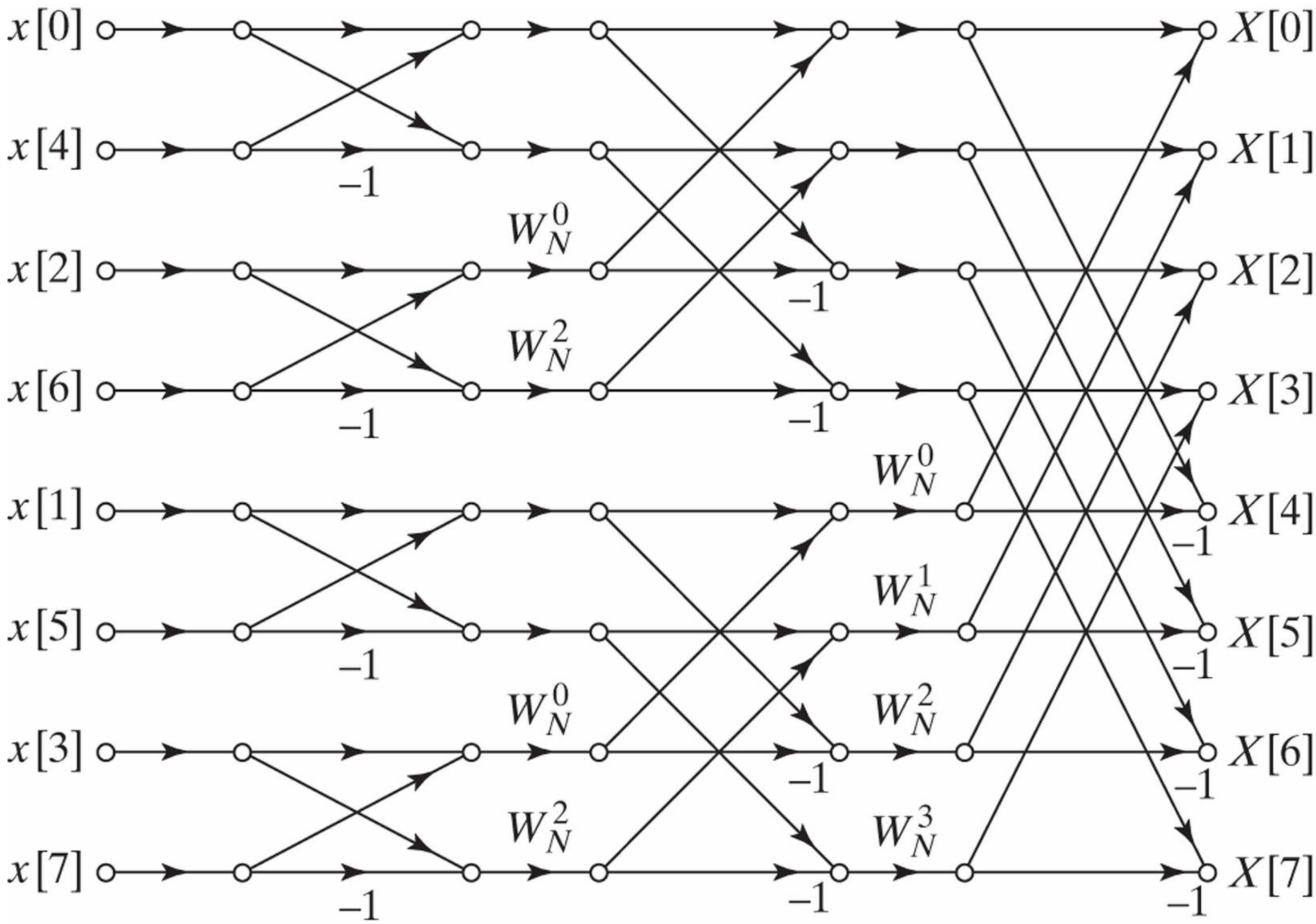


Figure 9.35 Butterfly computation for decimation-in-time.

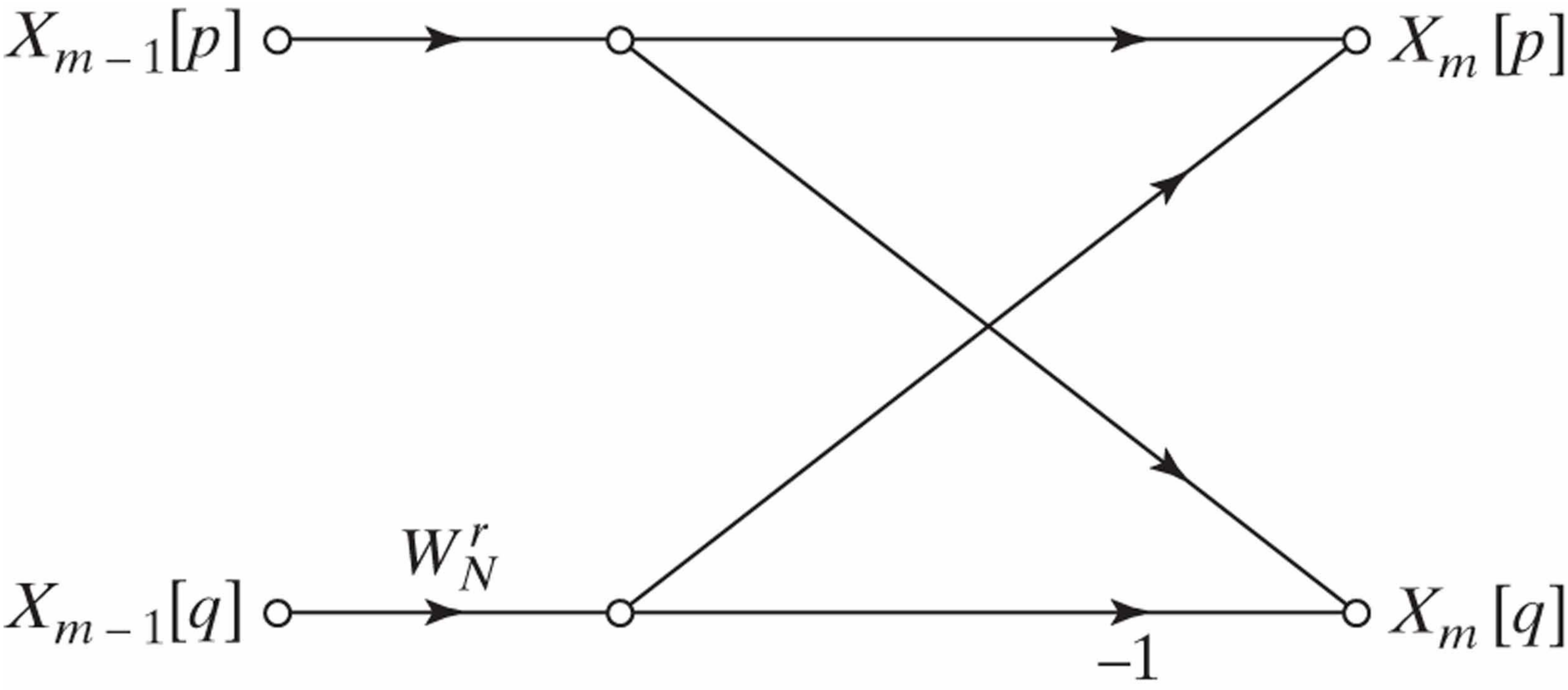


Figure 9.36 Linear-noise model for fixed-point round-off noise in a decimation-in-time butterfly computation.

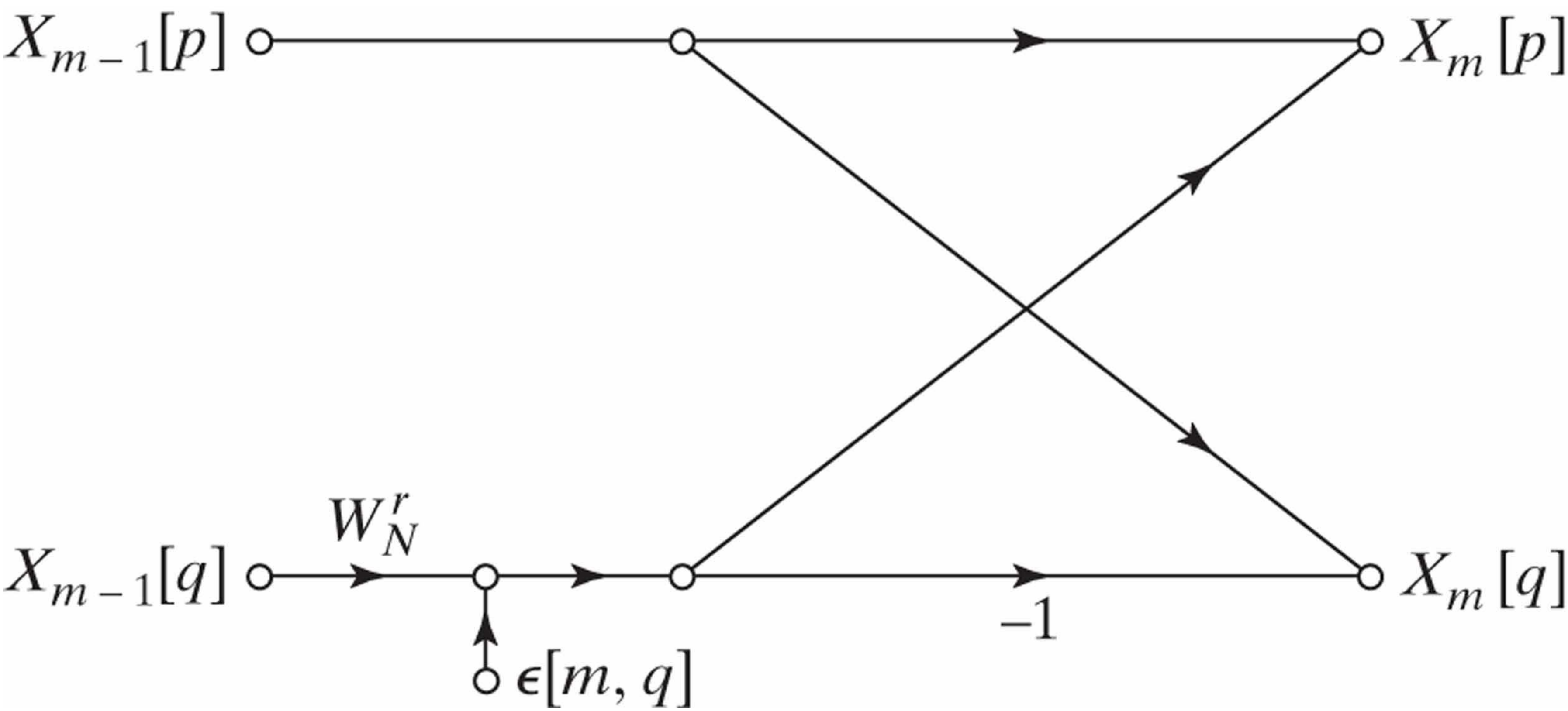


Figure 9.37 (a) Butterflies that affect $X[0]$; (b) butterflies that affect $X[2]$.

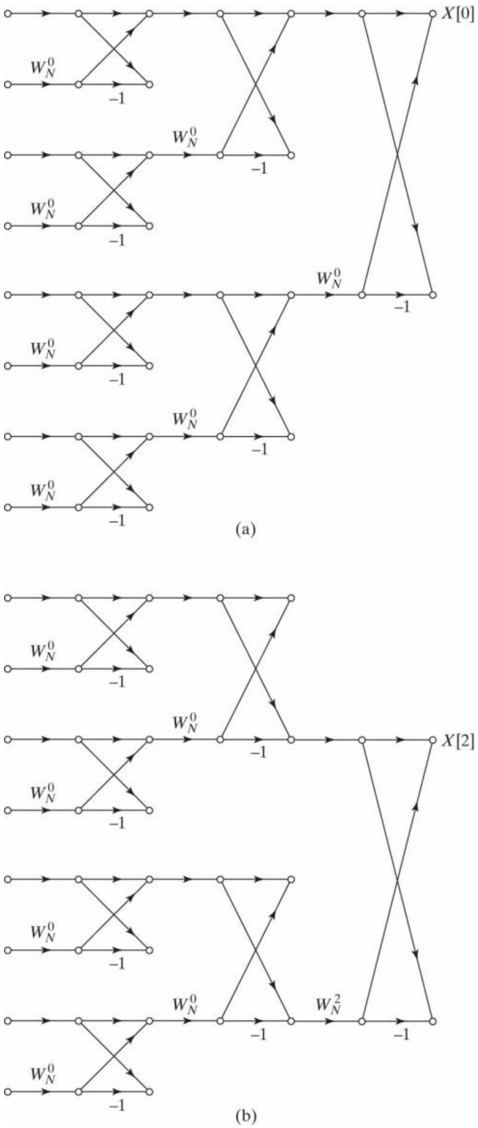


Figure 9.38 Butterfly showing scaling multipliers and associated fixed-point round-off noise.

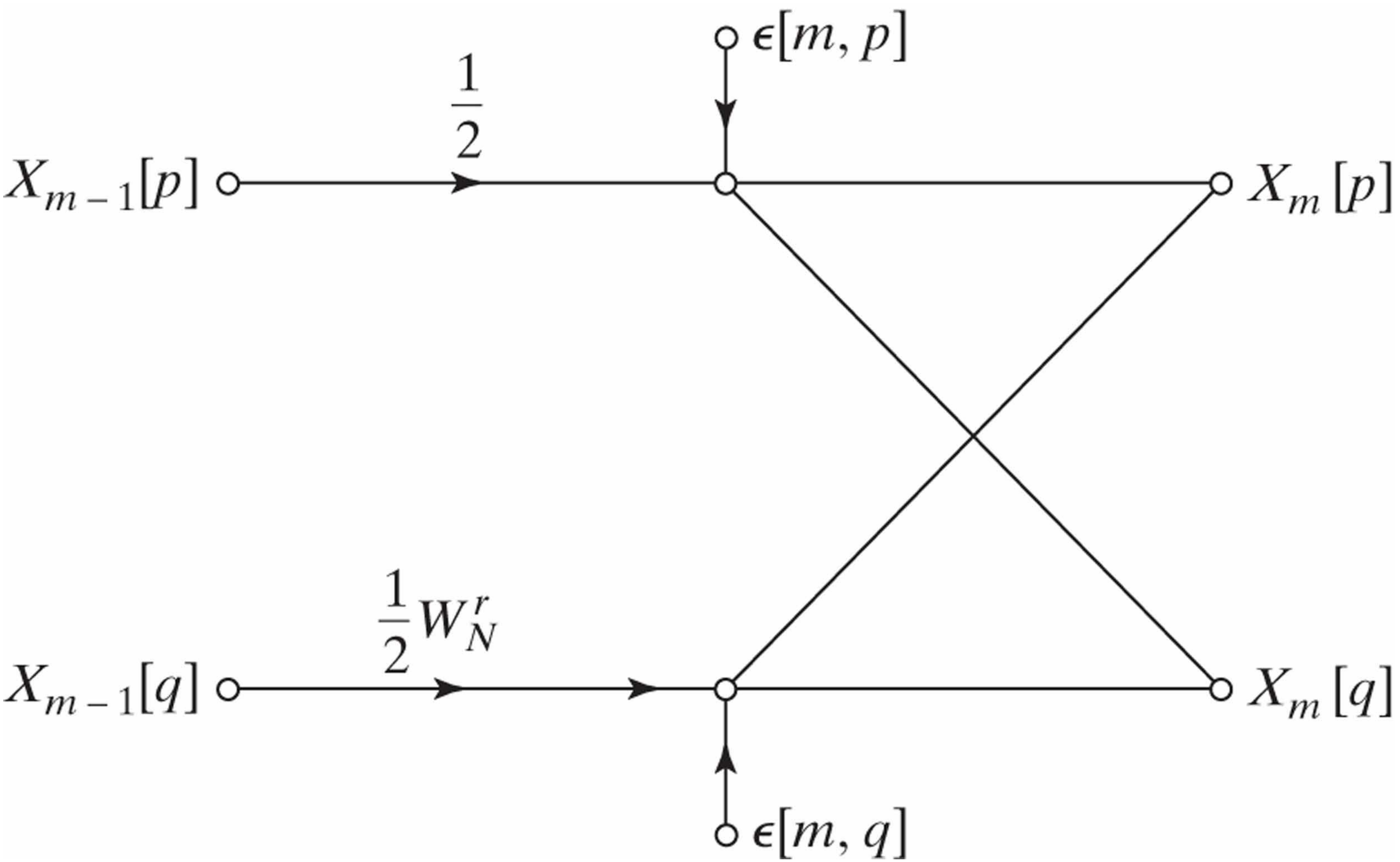


Figure P9.4

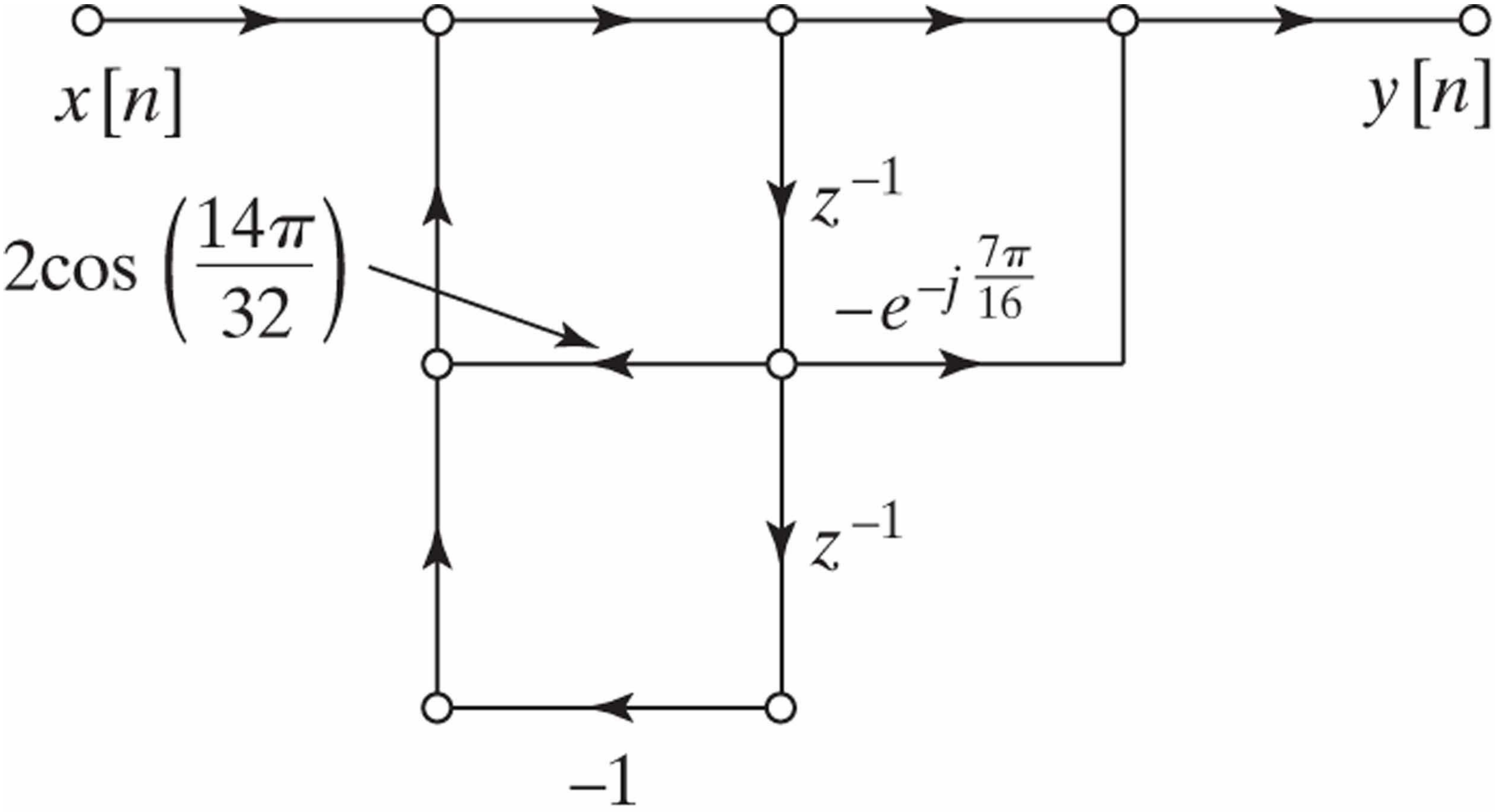


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Figure P9.5

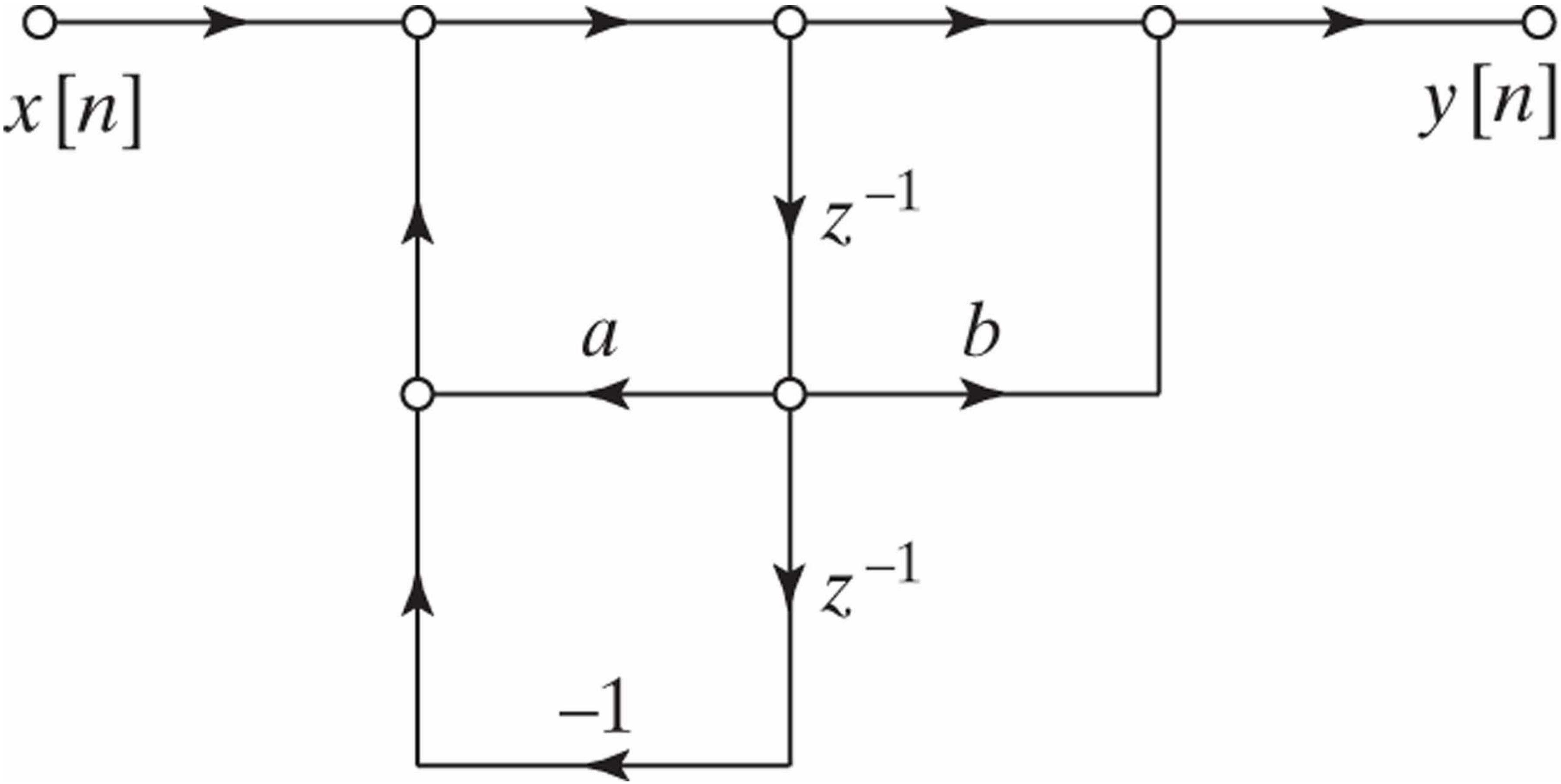
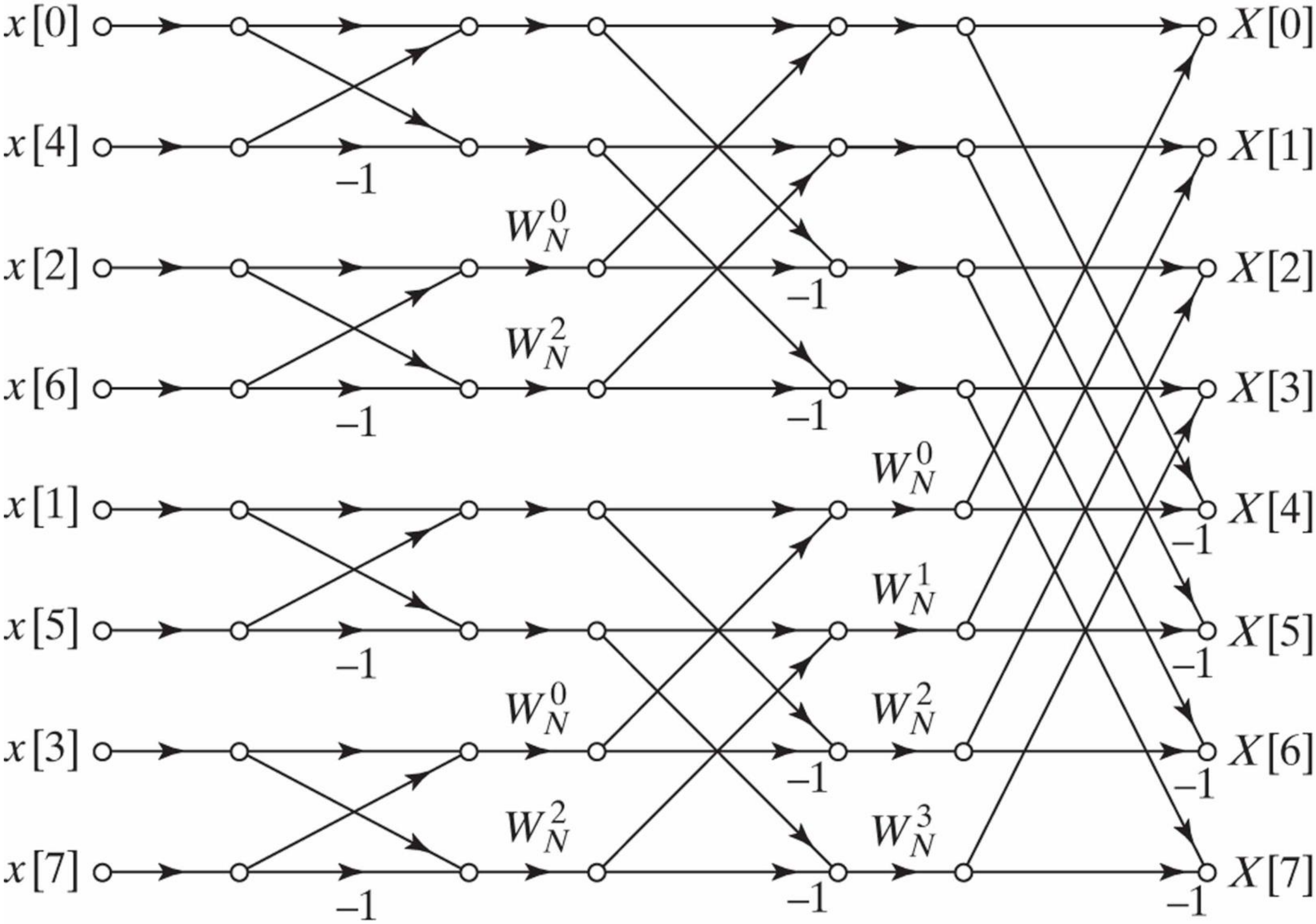


Figure P9.6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x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]
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W_N^0
W_N^2
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W_N^2
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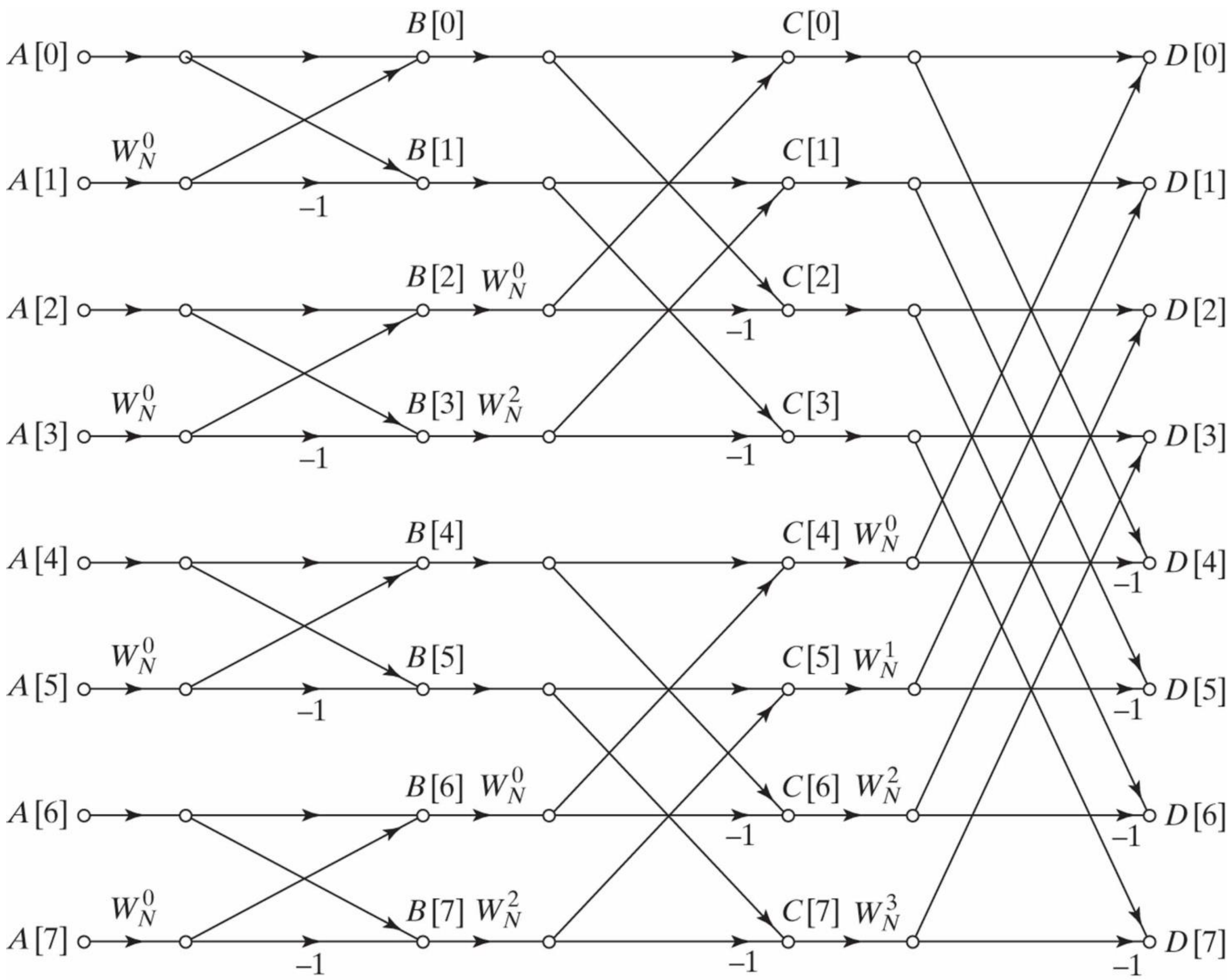
W_N^0
W_N^1
W_N^2
W_N^3
–1
–1
–1
–1
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]

Figure P9.7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A[2]
A[3]
A[4]
A[5]
A[6]
A[7]
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W_N^0
W_N^0
W_N^0
–1
–1
–1
–1
B[0]
B[1]
B[2] W_N^0
B[3] W_N^2
B[4]
B[5]
B[6] W_N^0
B[7] W_N^2
–1
–1
–1
–1
C[0]
C[1]
C[2]
C[3]
C[4] W_N^0
C[5] W_N^1
C[6] W_N^2
C[7] W_N^3
–1
–1
–1
–1
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D[1]
D[2]
D[3]
D[4]
D[5]
D[6]
D[7]

Figure P9.9

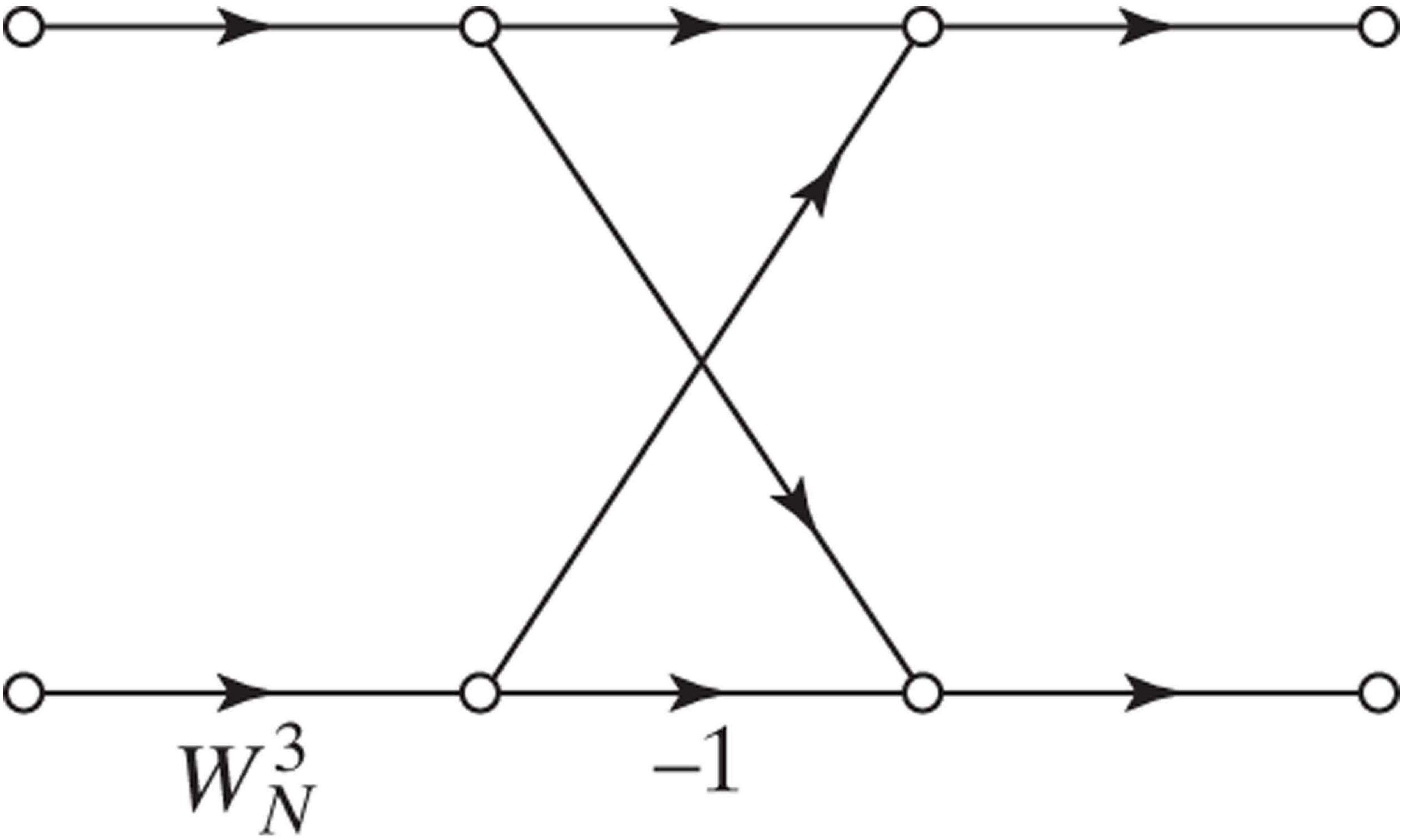

Figure P9.10

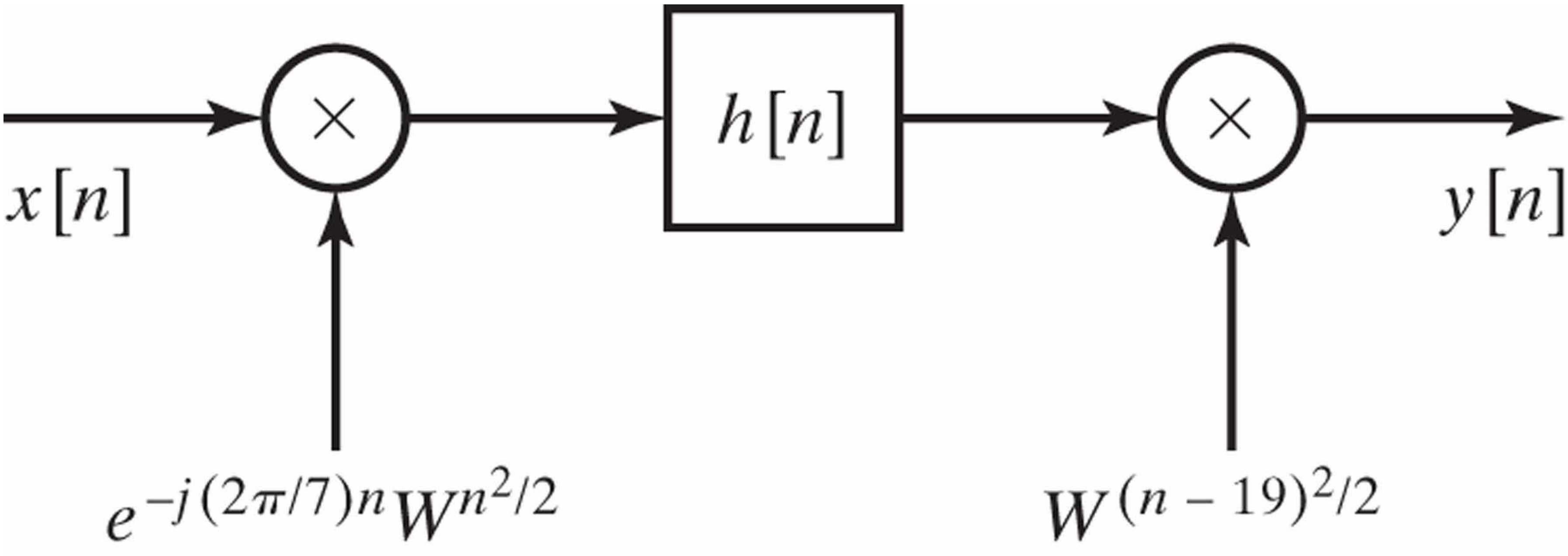

Figure P9.12

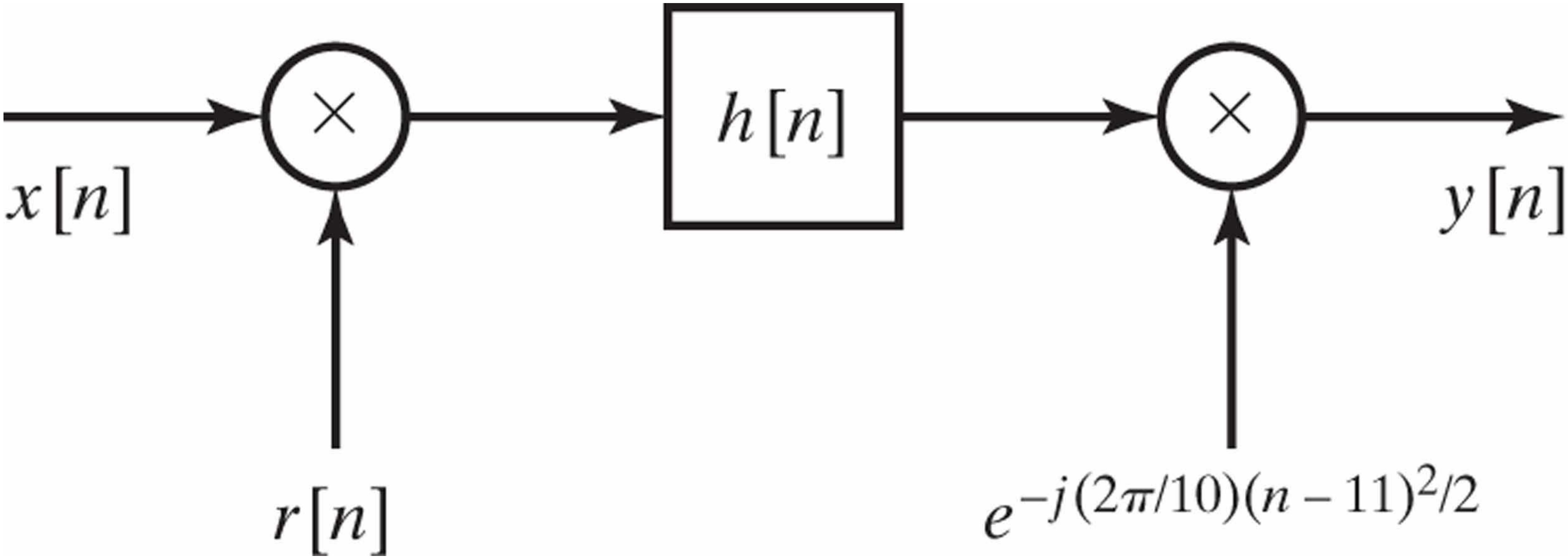

Figure P9.15

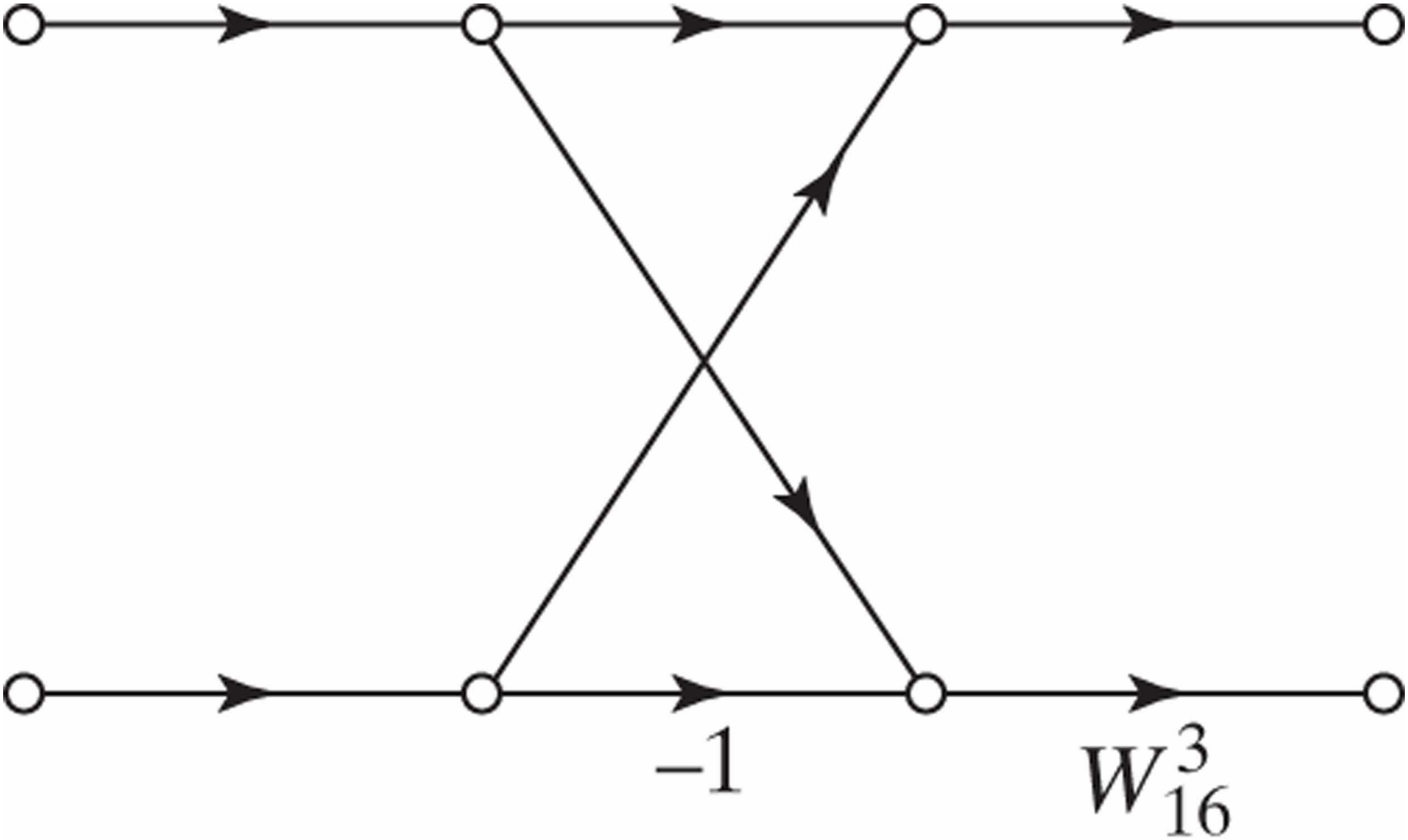

Figure P9.16

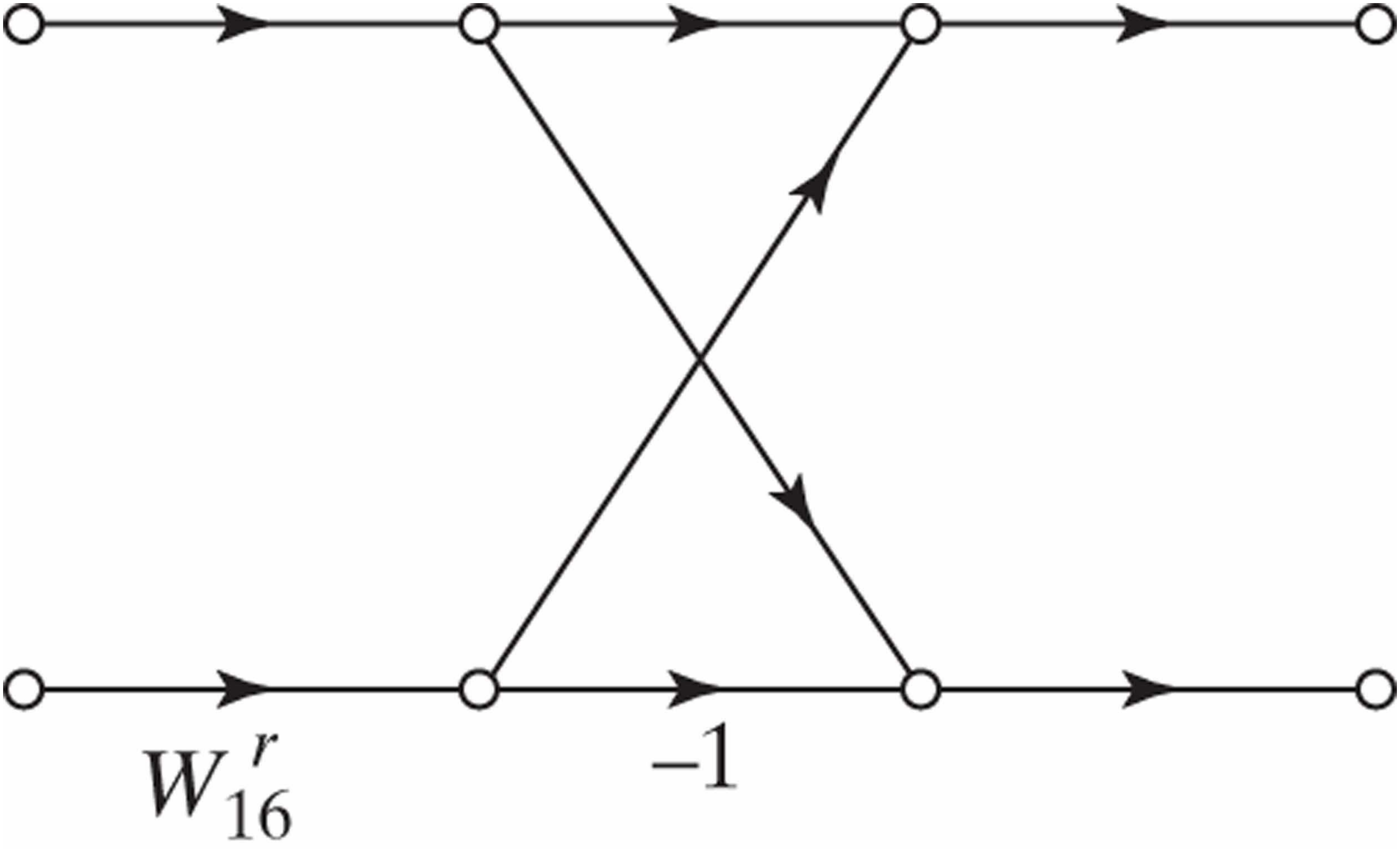

Figure P9.18

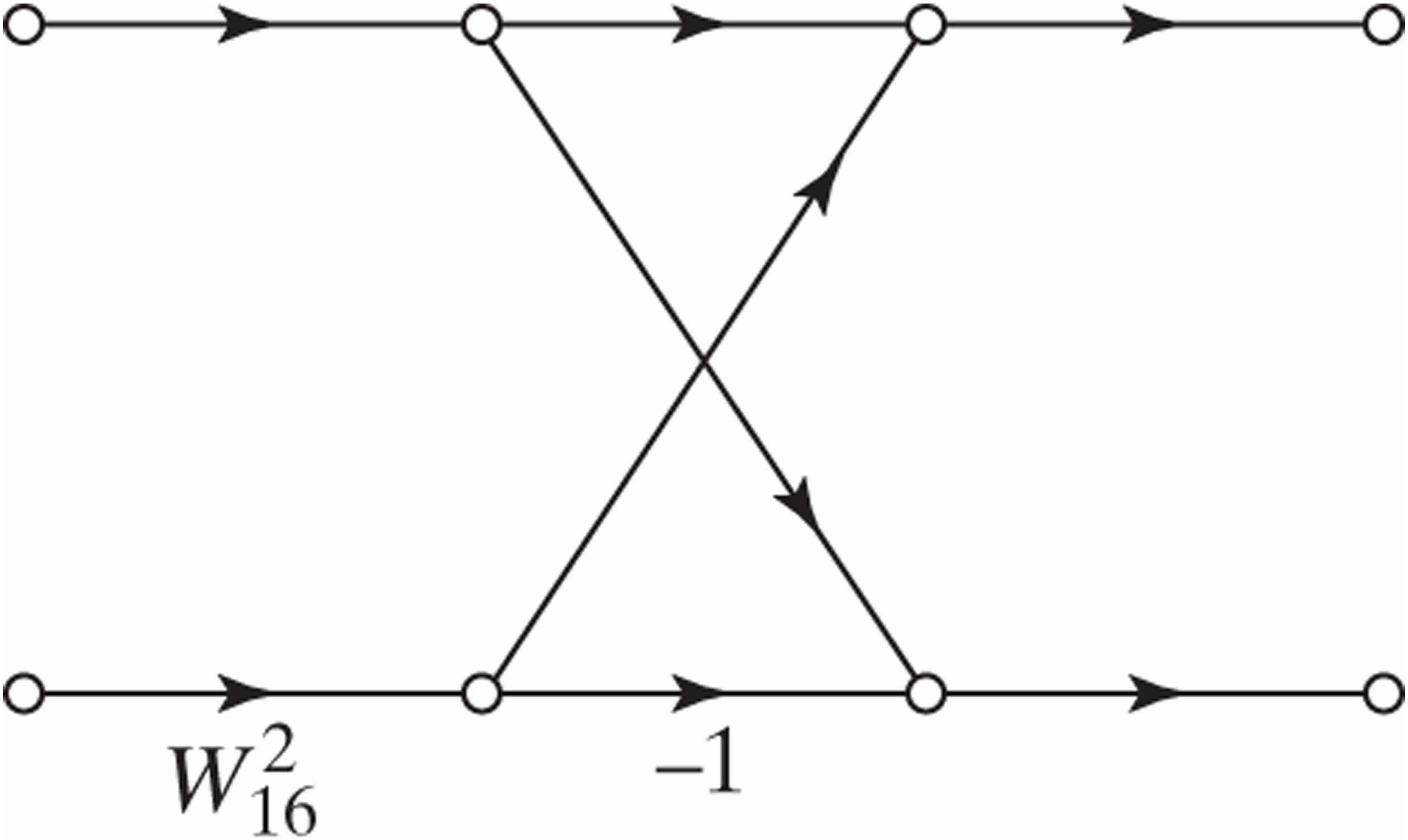

Figure P9.21-1

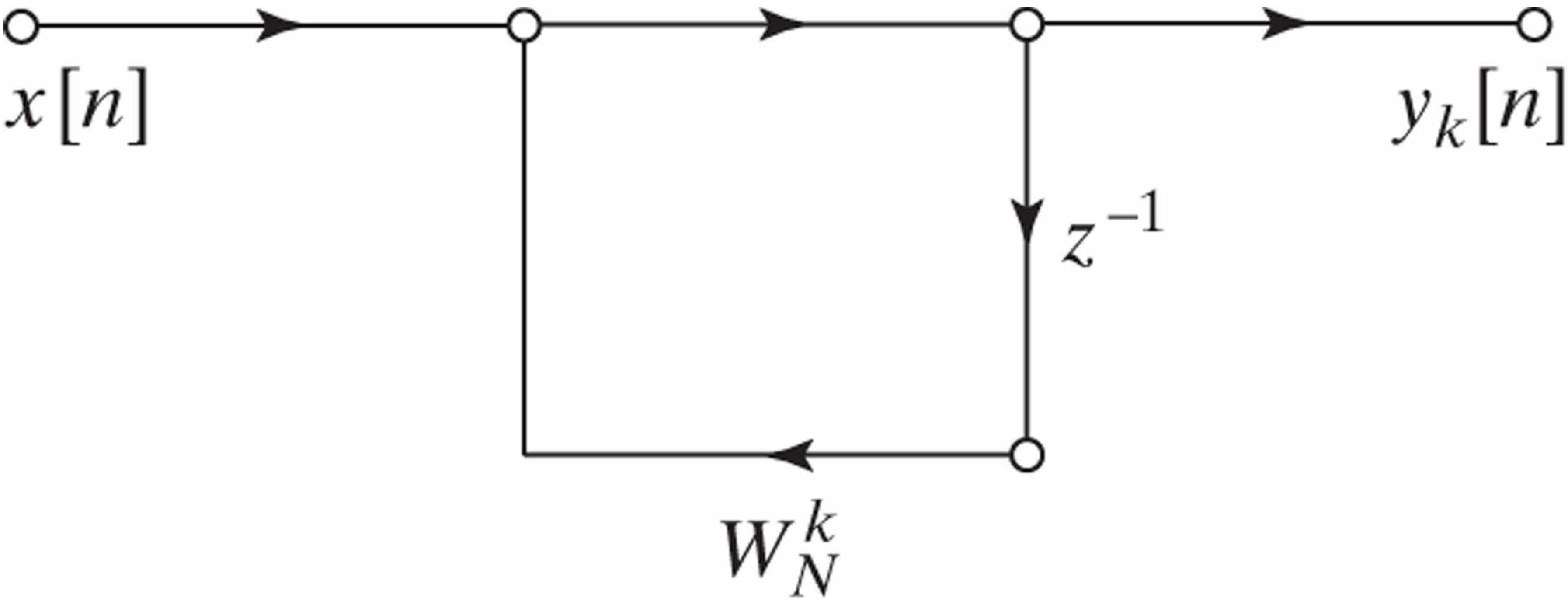

Figure P9.21-2

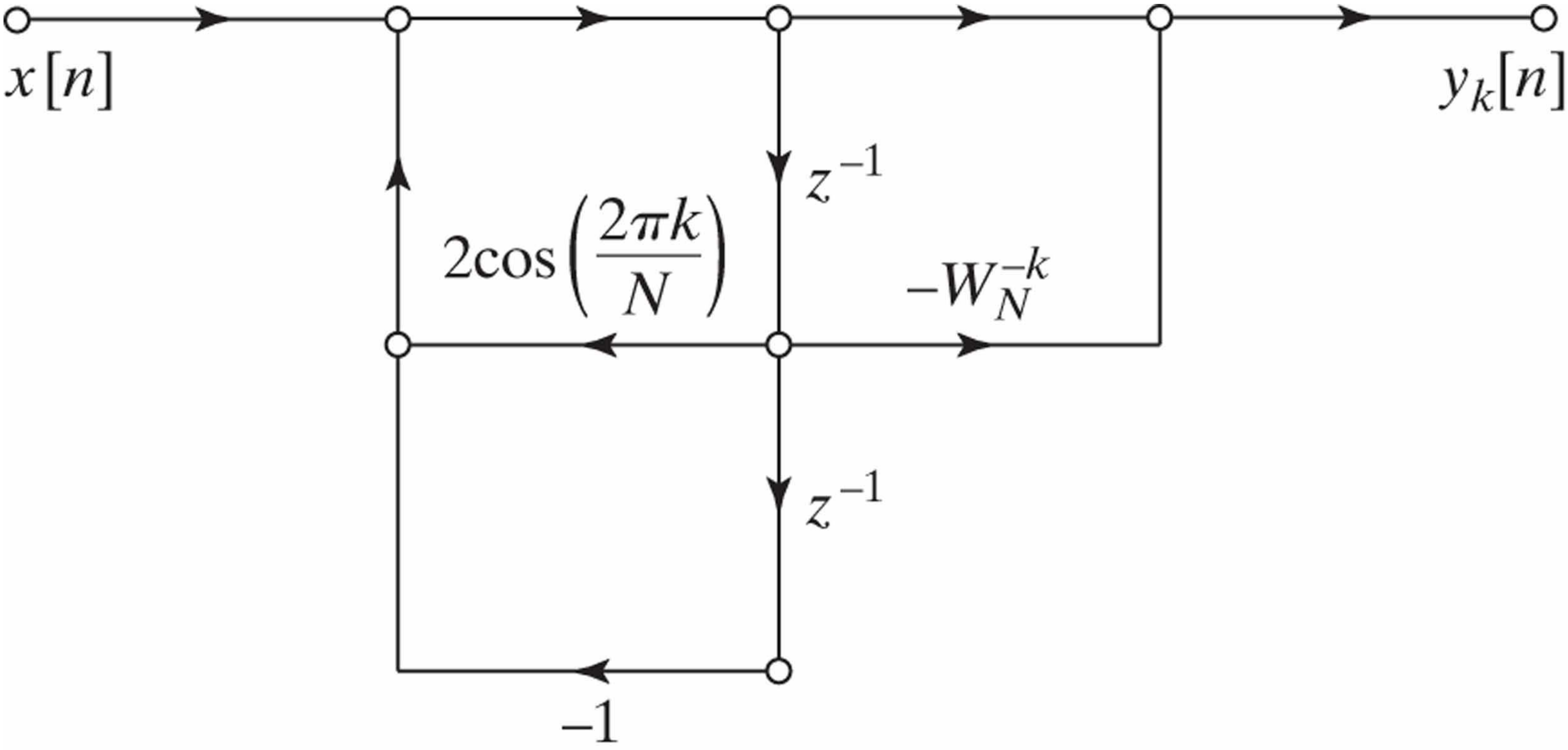

Figure P9.22

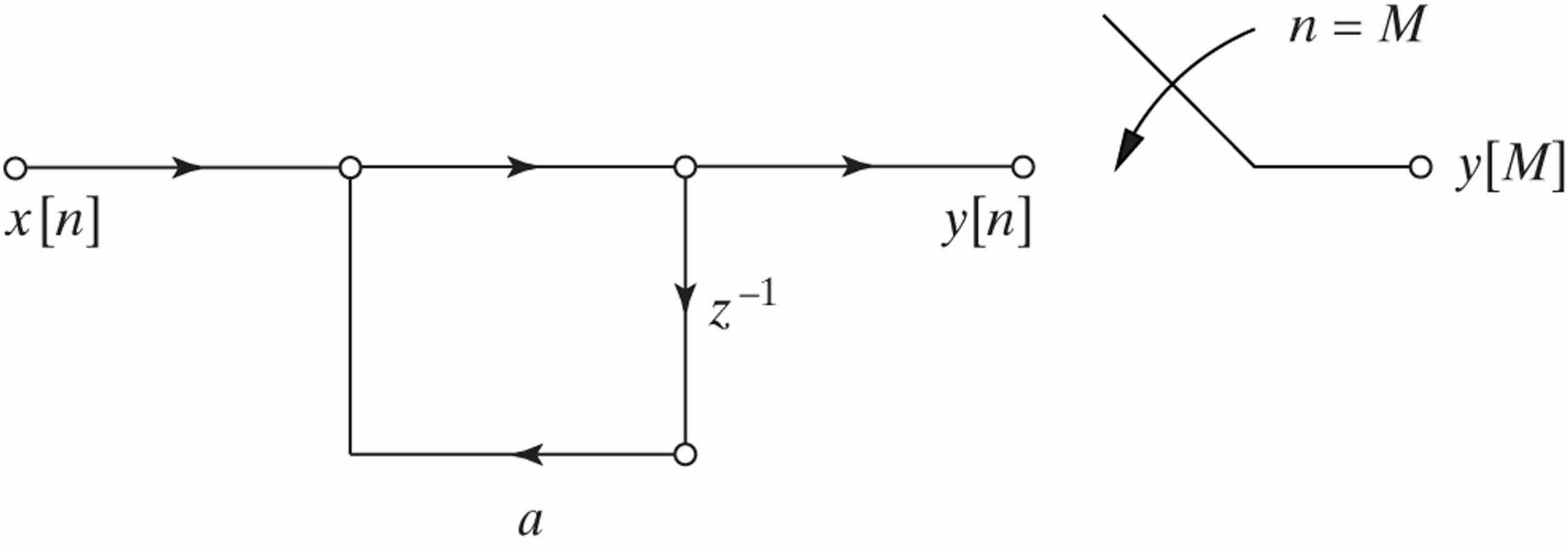

Table 9.1

TABLE 9.1

Module	Per-Unit Cost
8-point DFT	$1
8-point IDFT	$1
adder	$10
multiplier	$100

Figure P9.30

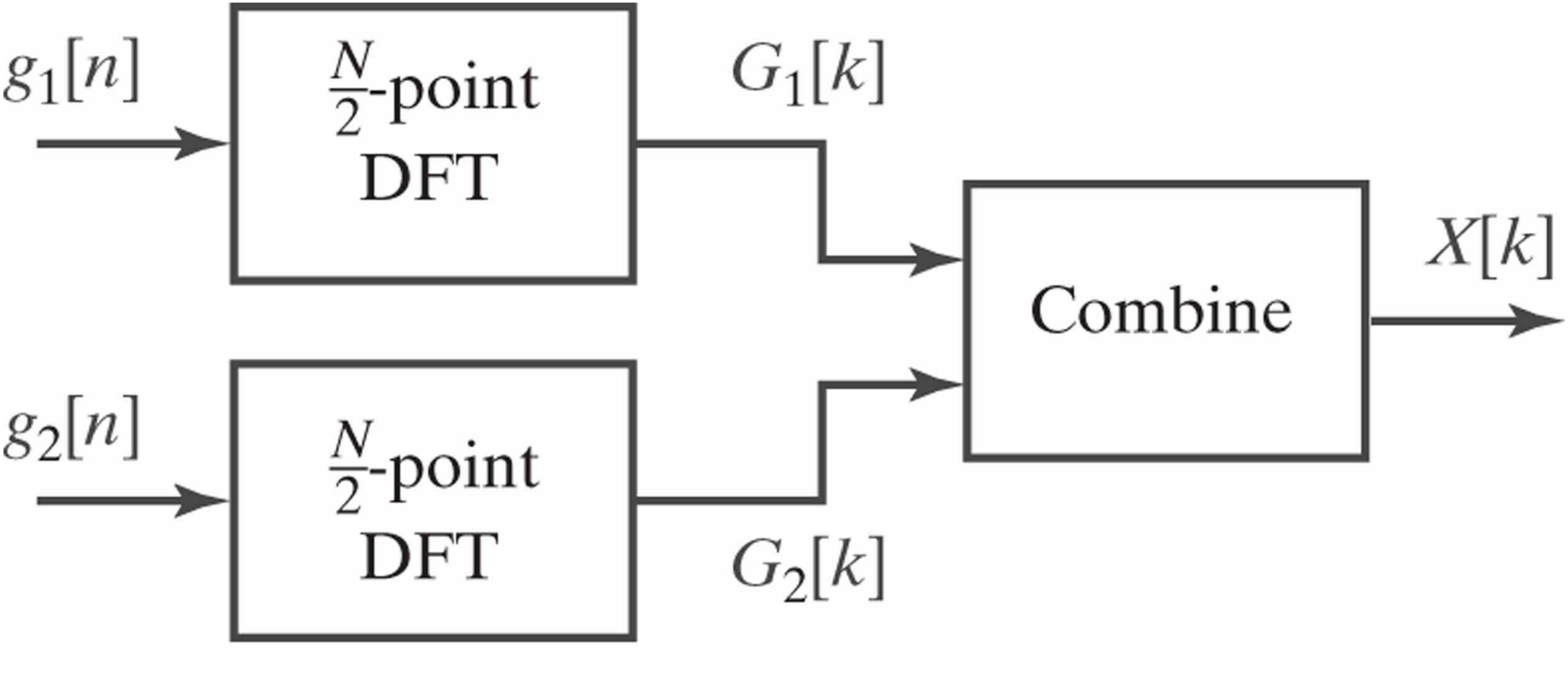

Figure P9.31-1

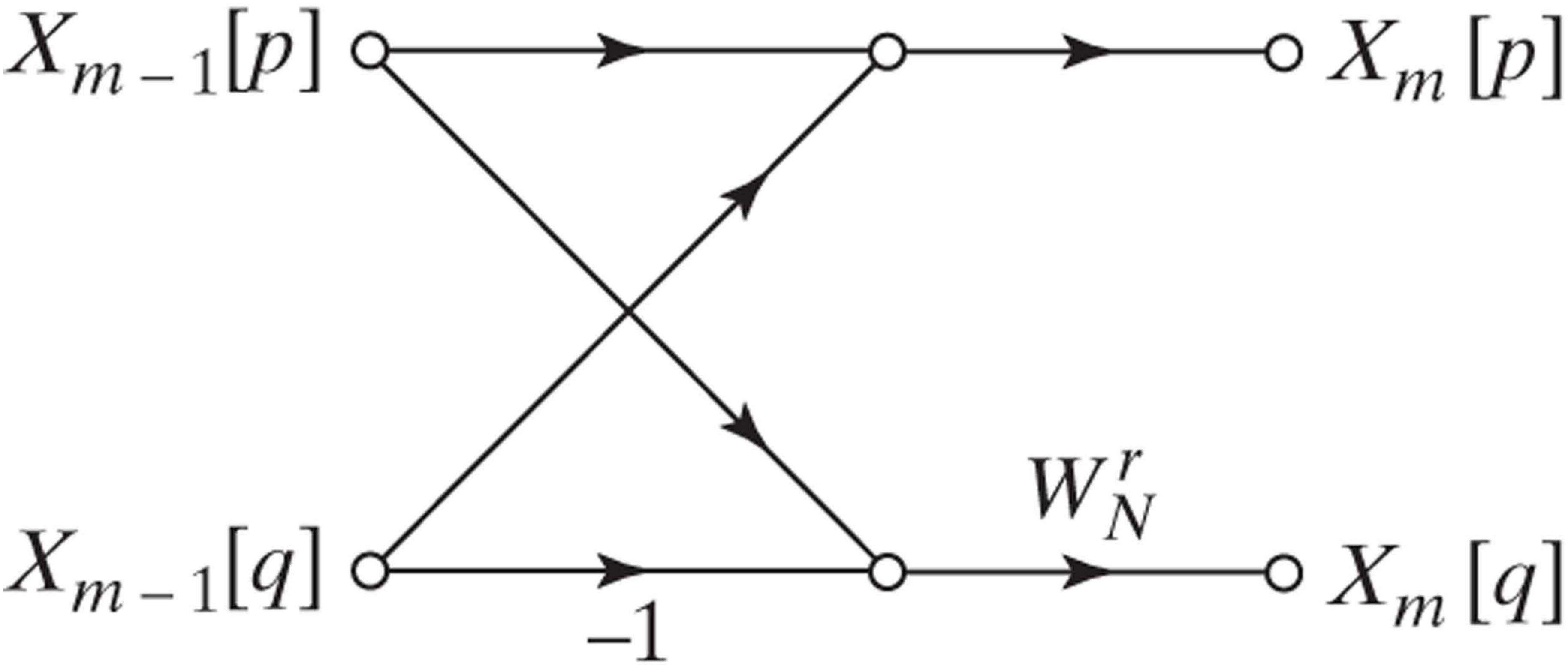

Figure P9.31-2

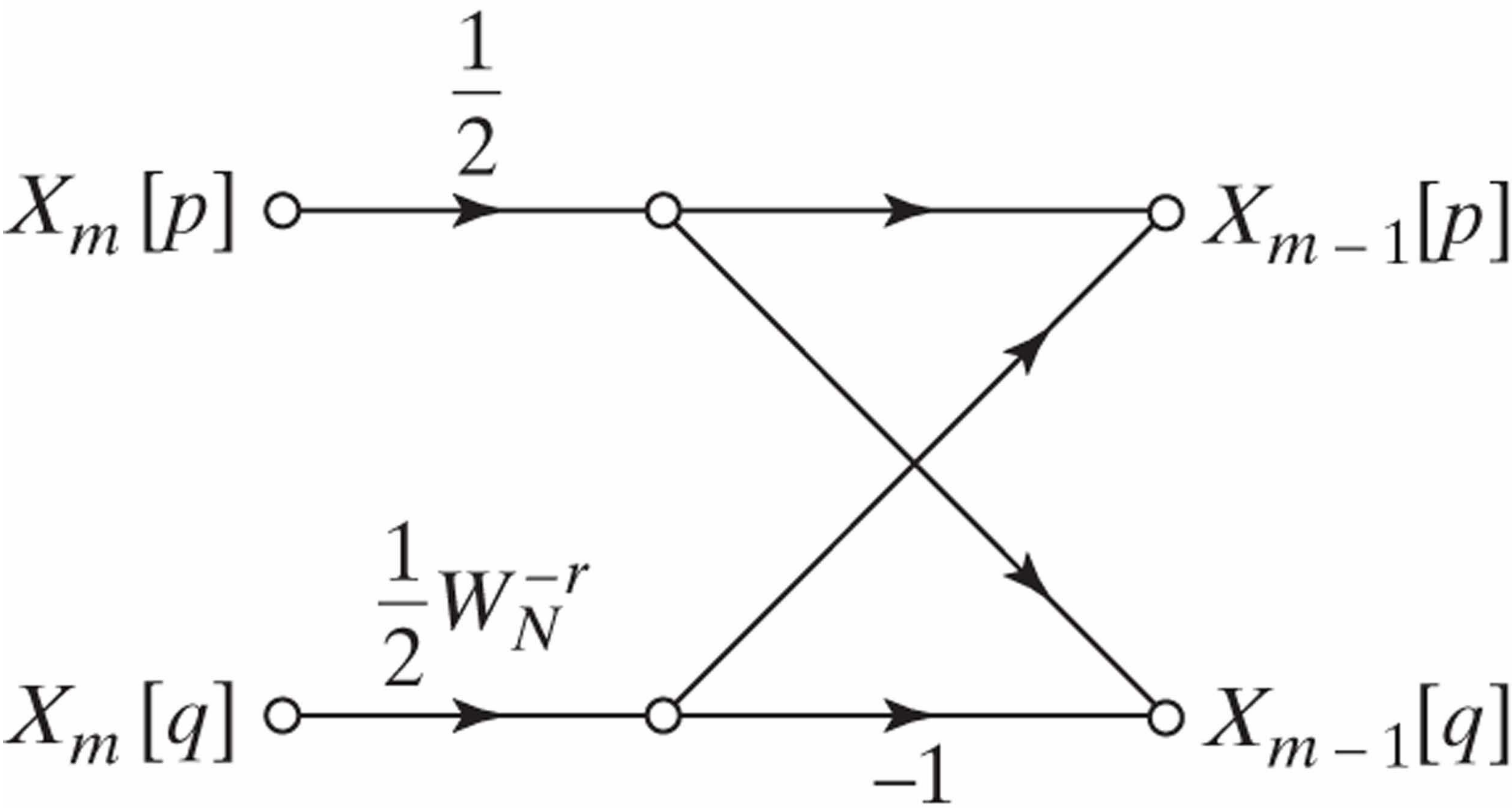

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Figure P9.32-1

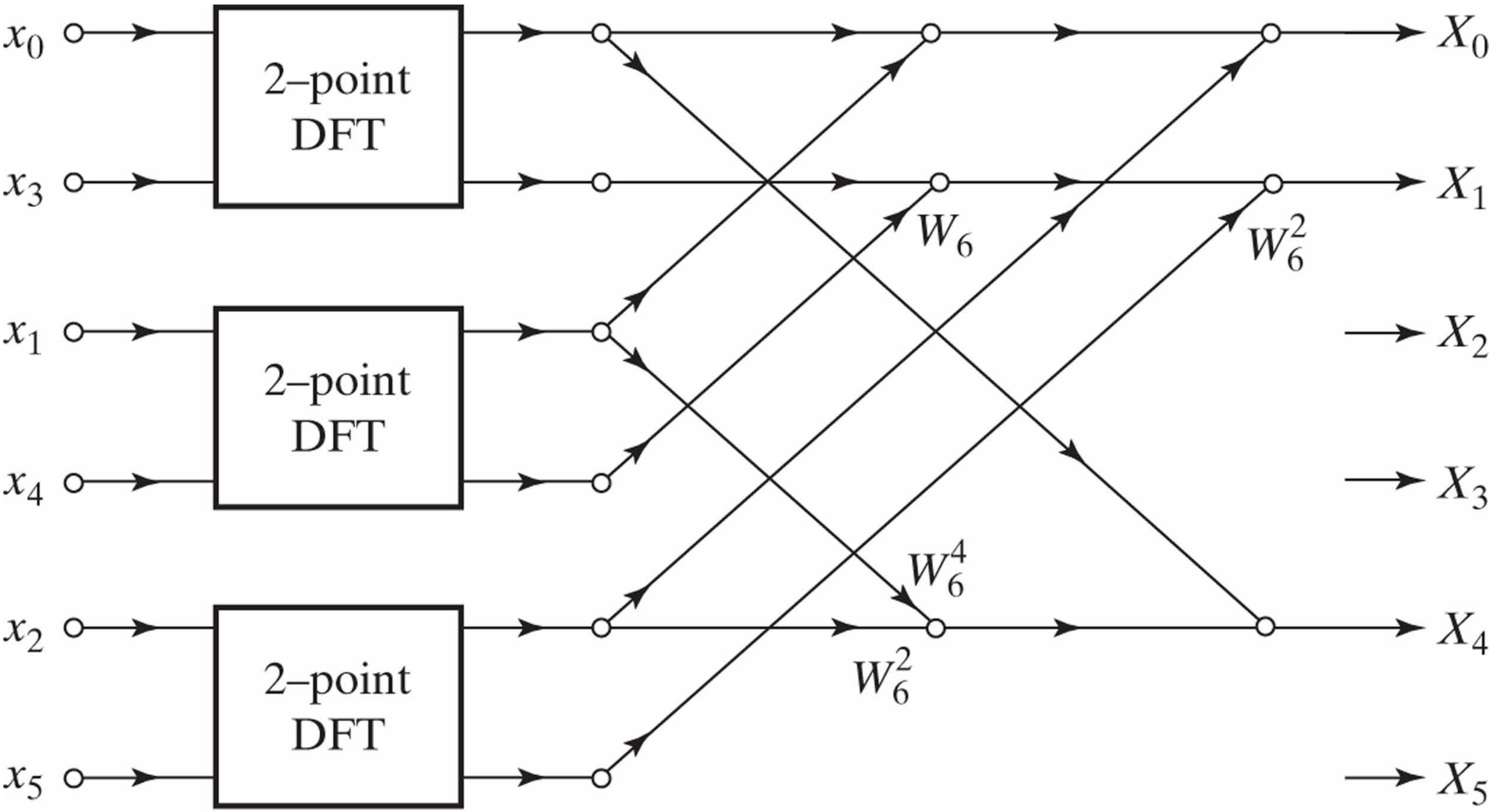


Figure P9.32-2

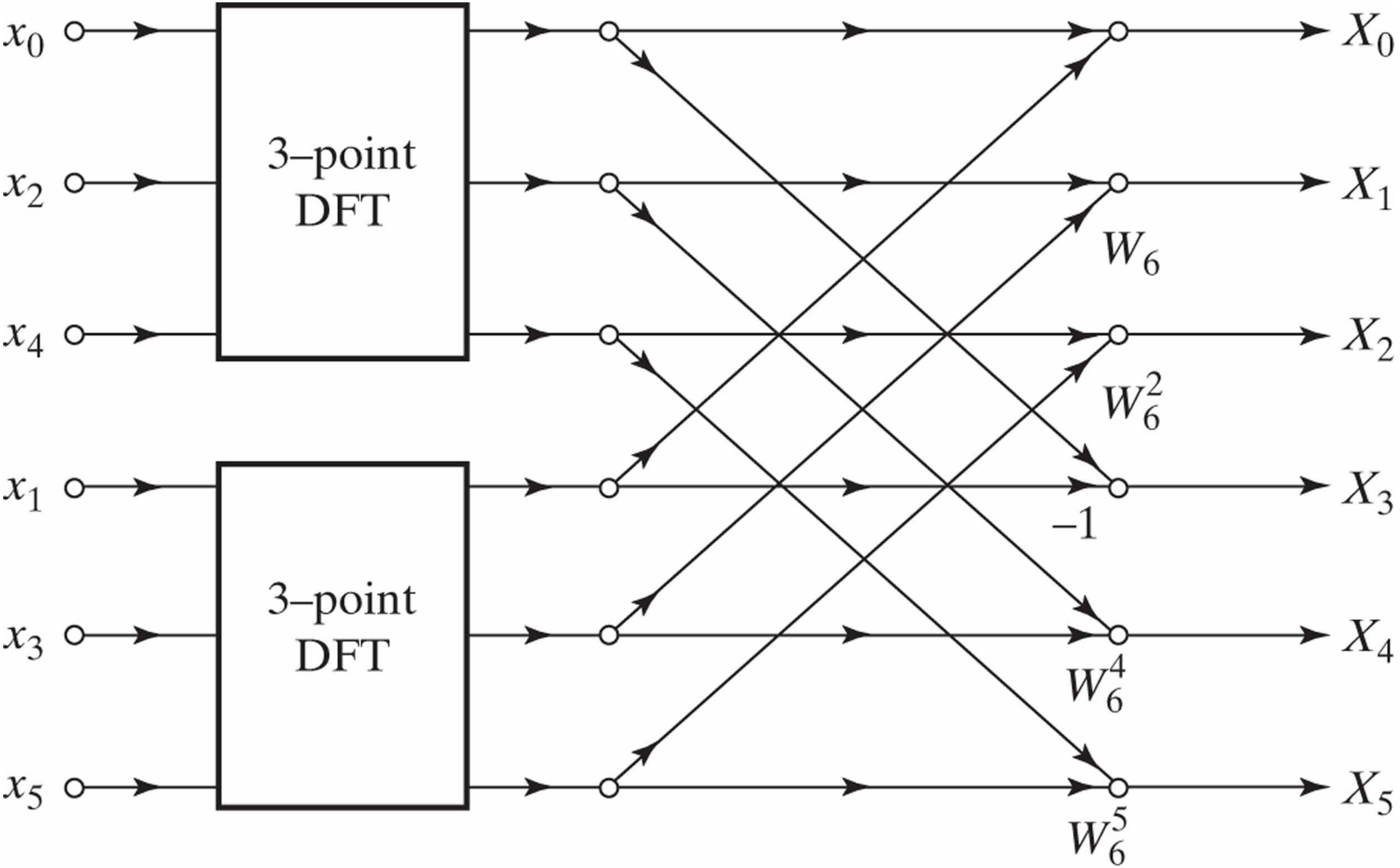


Figure P9.34-1

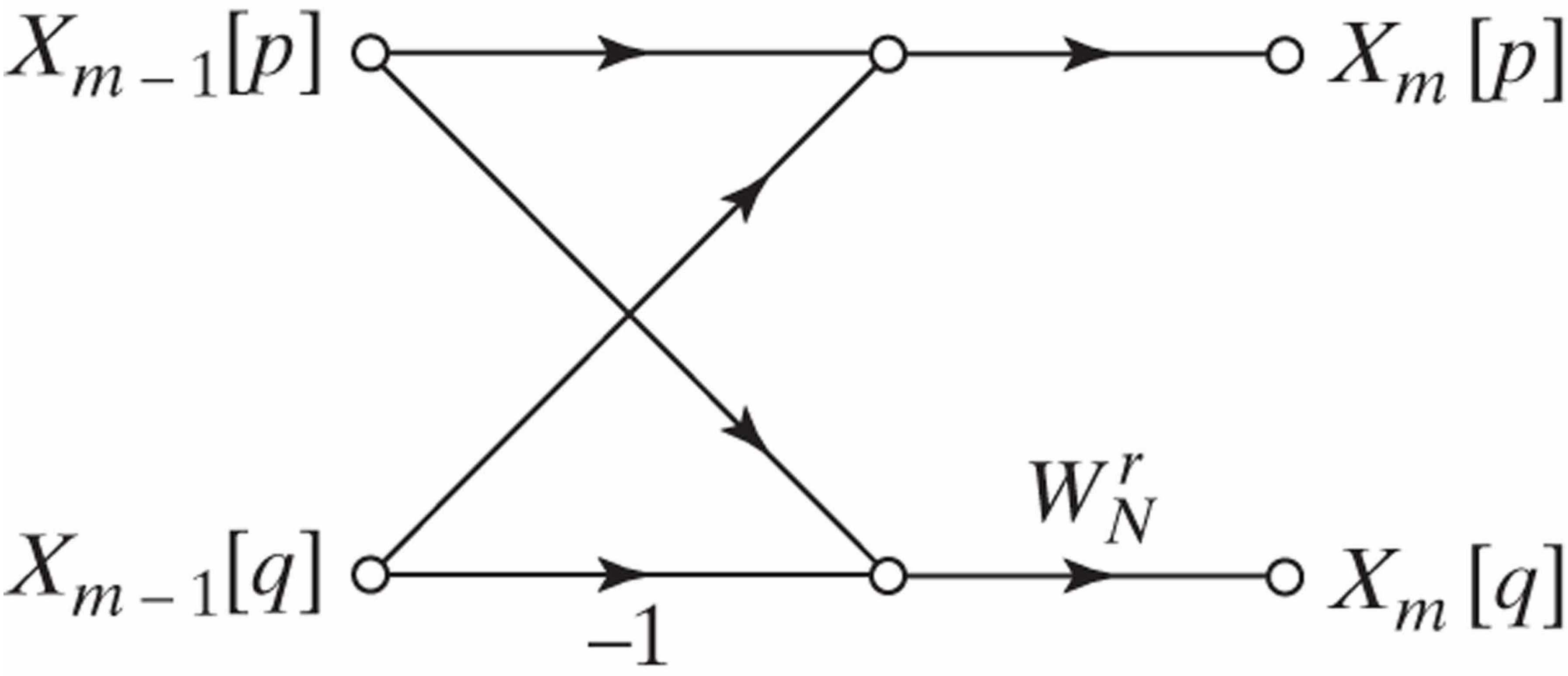


Figure P9.34-2

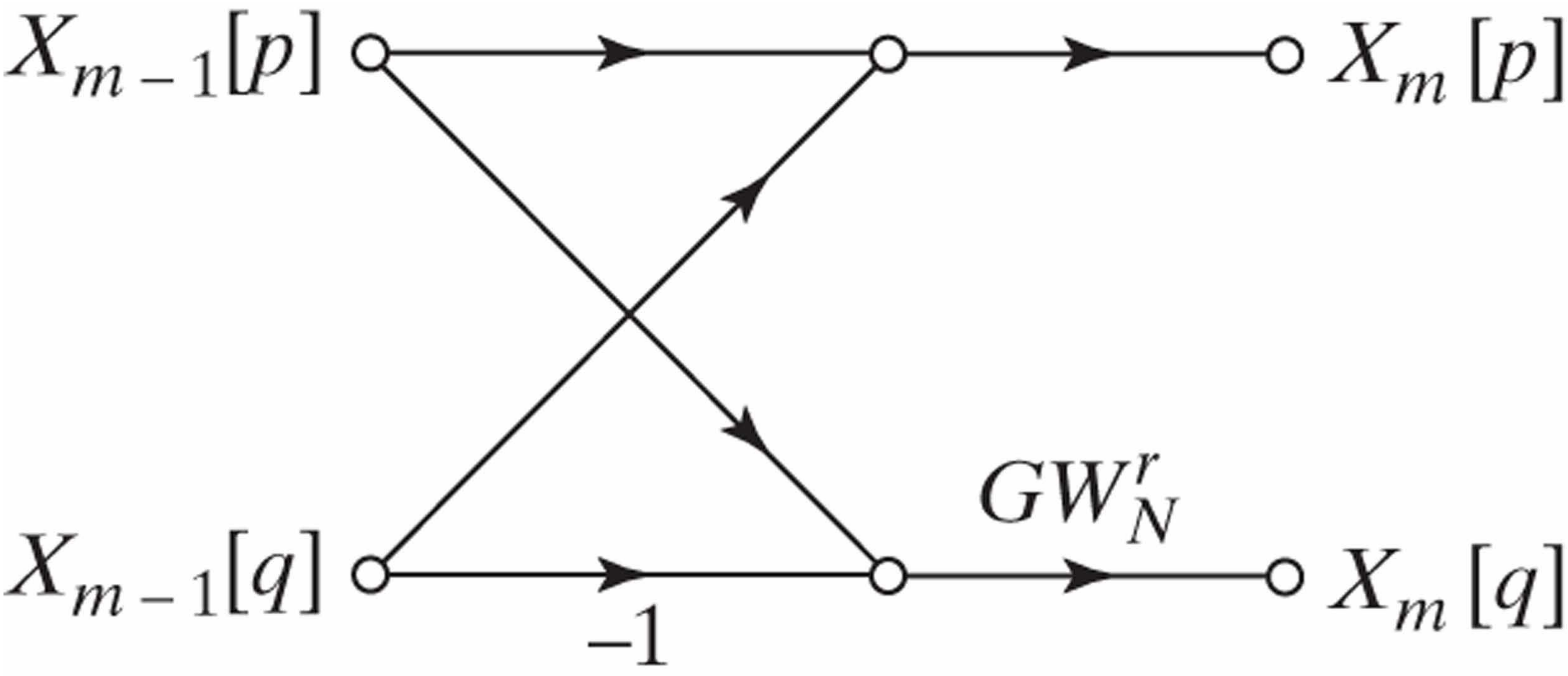


Figure P9.34-3

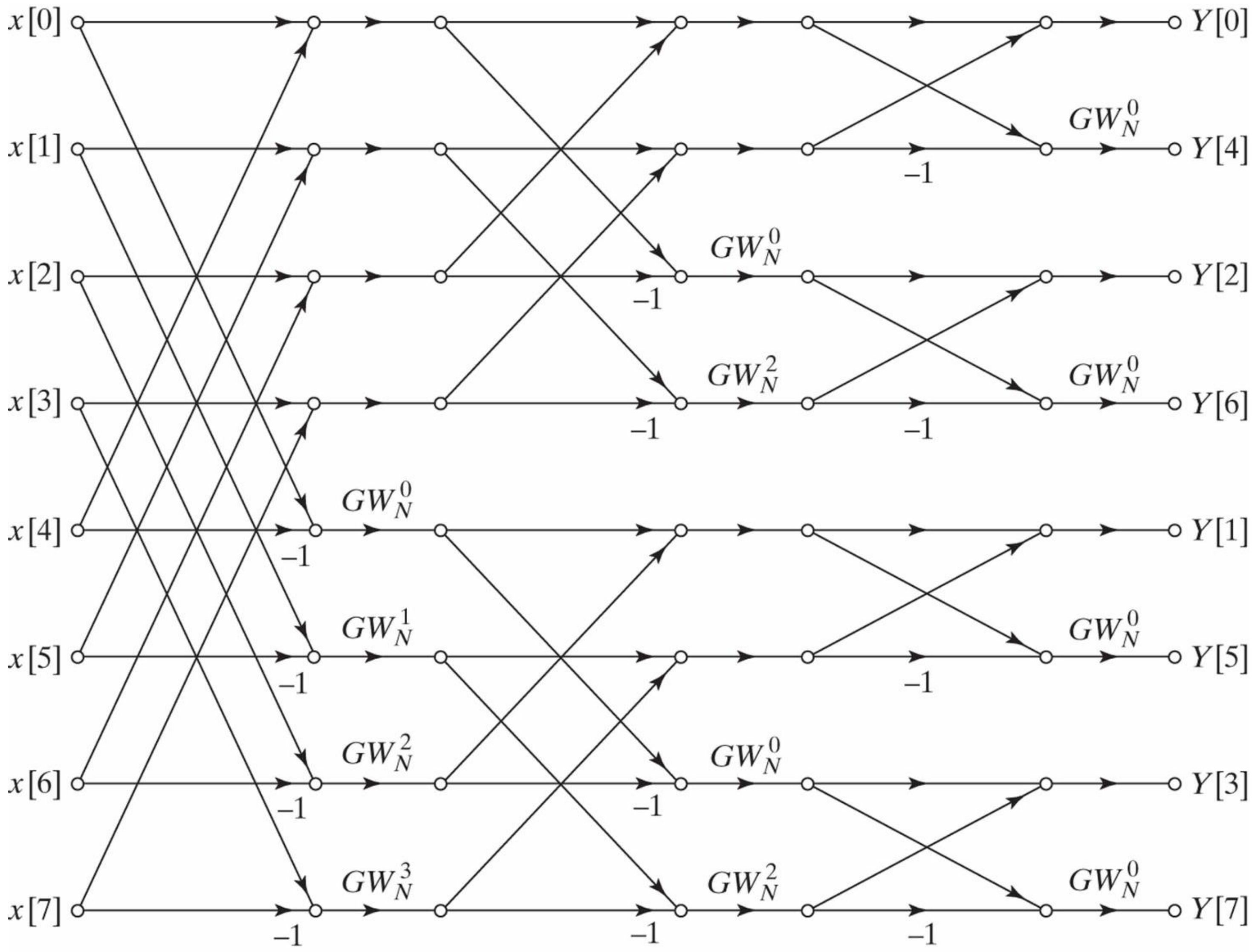


Figure P9.36

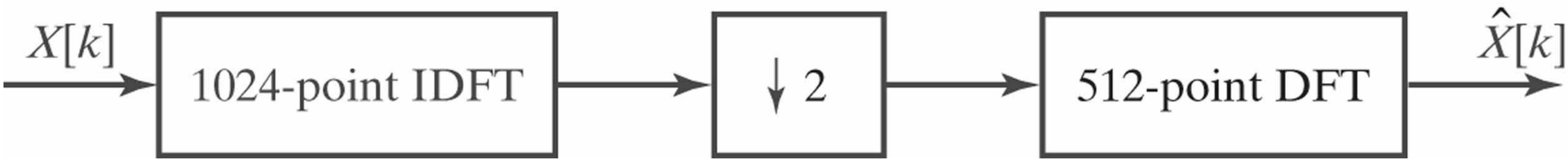


Figure P9.40

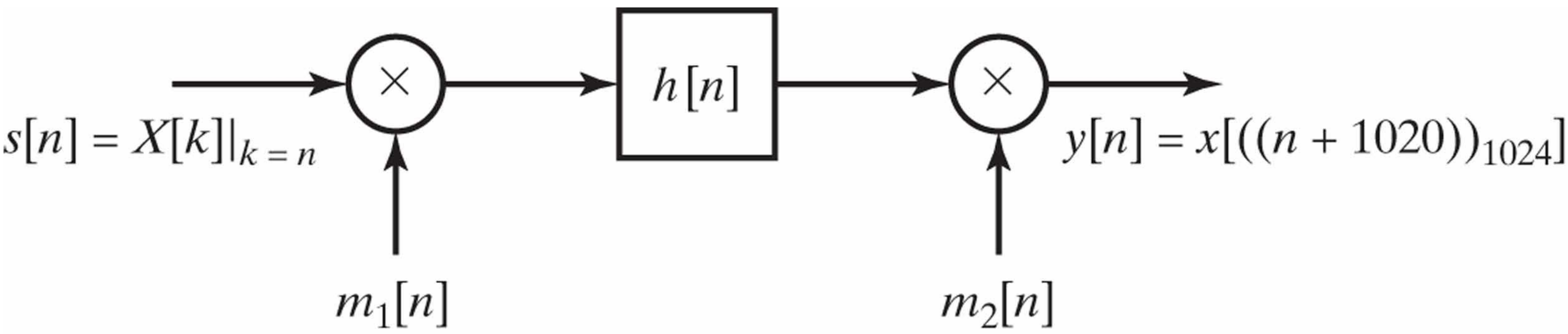


Figure P9.42-1

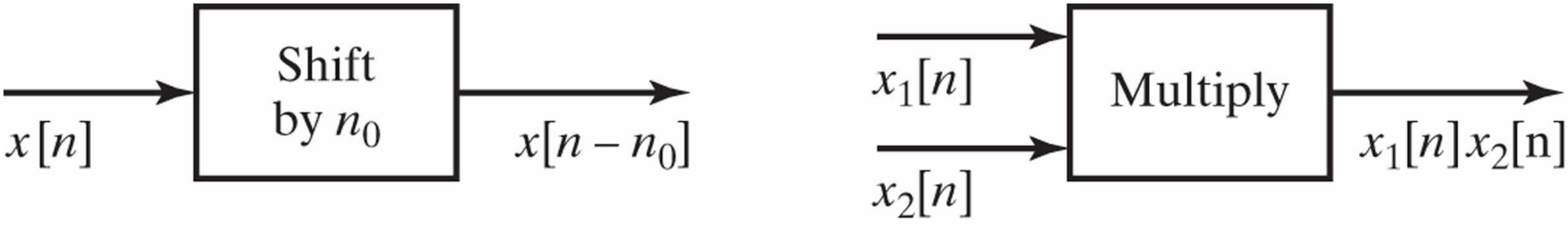


Figure P9.42-2

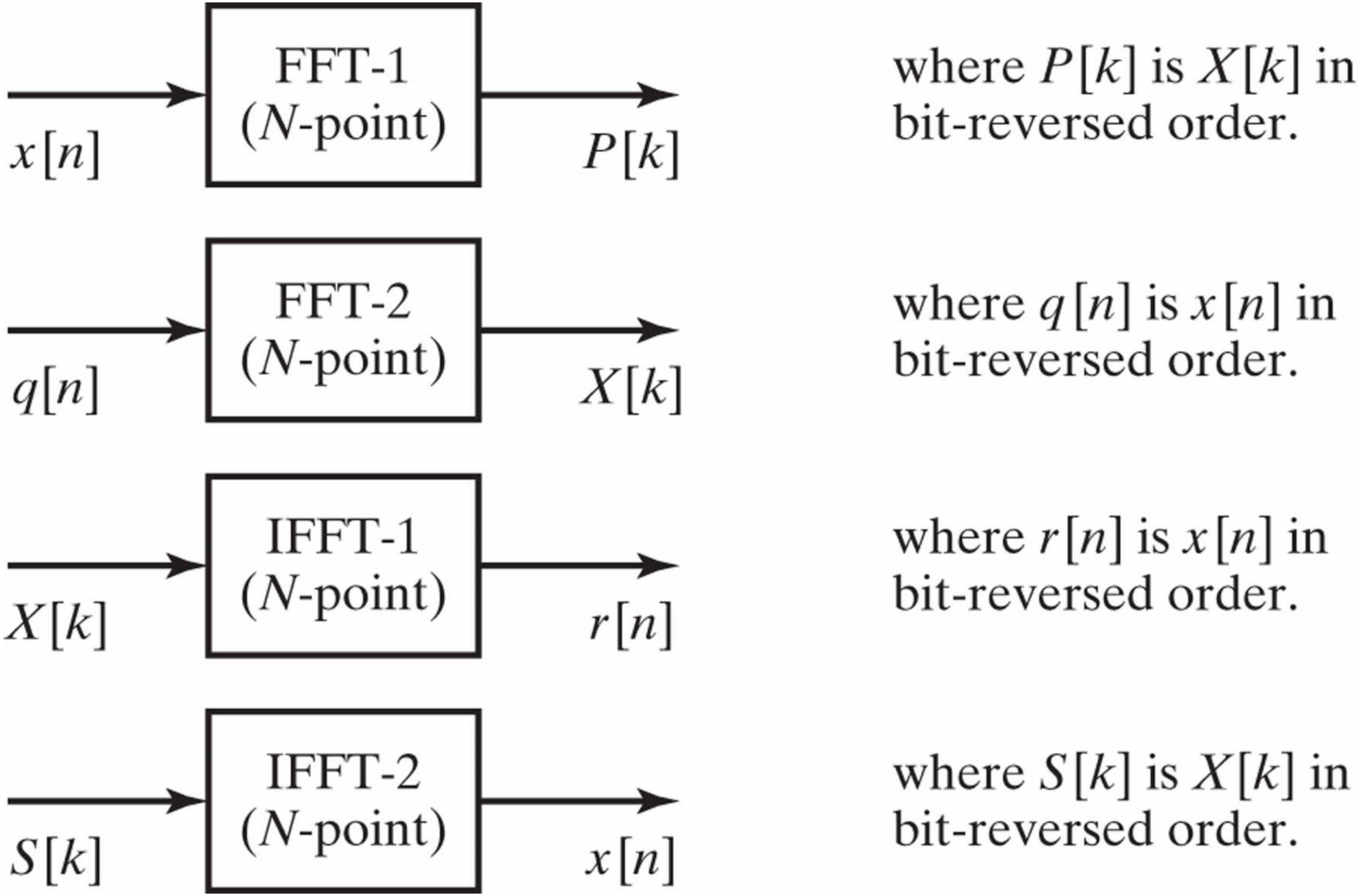


Figure P9.43

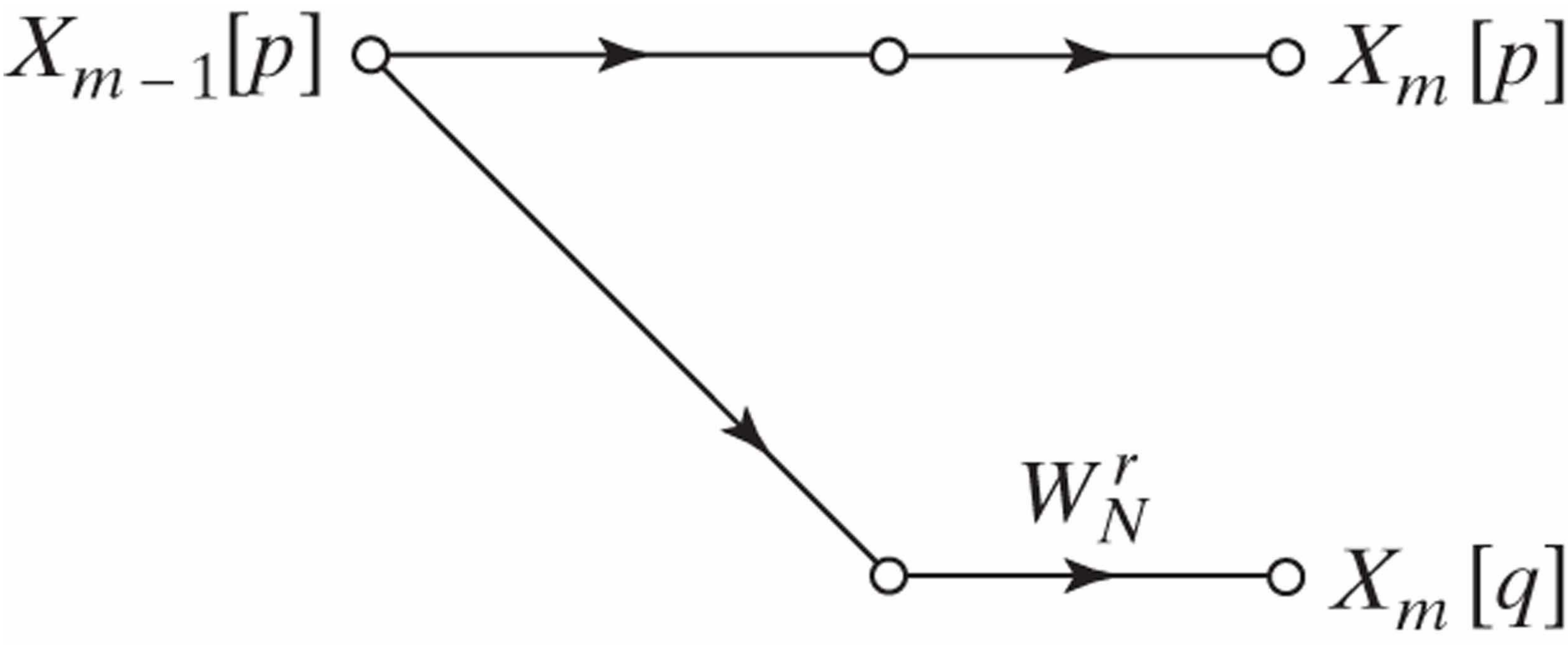
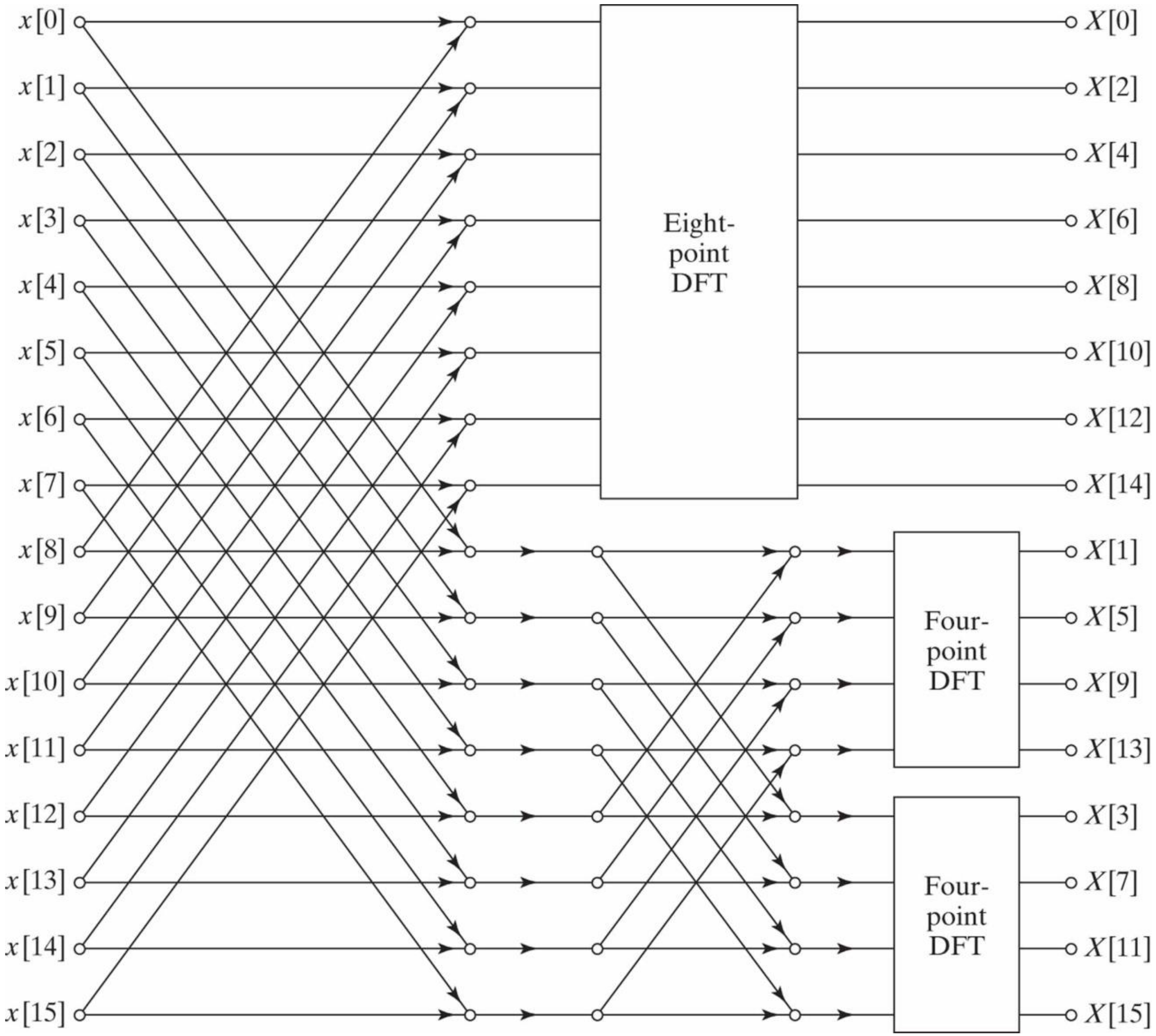


Figure P9.45



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Four-point DFT
Four-point DFT
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Figure P9.48

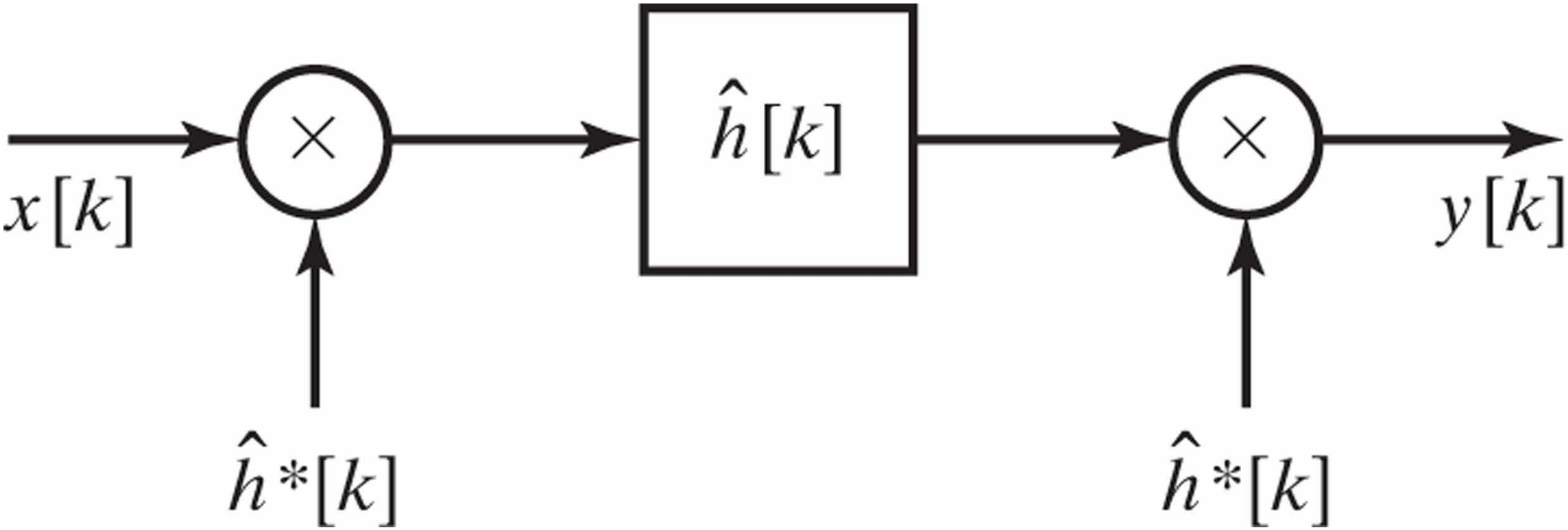


Figure P9.49

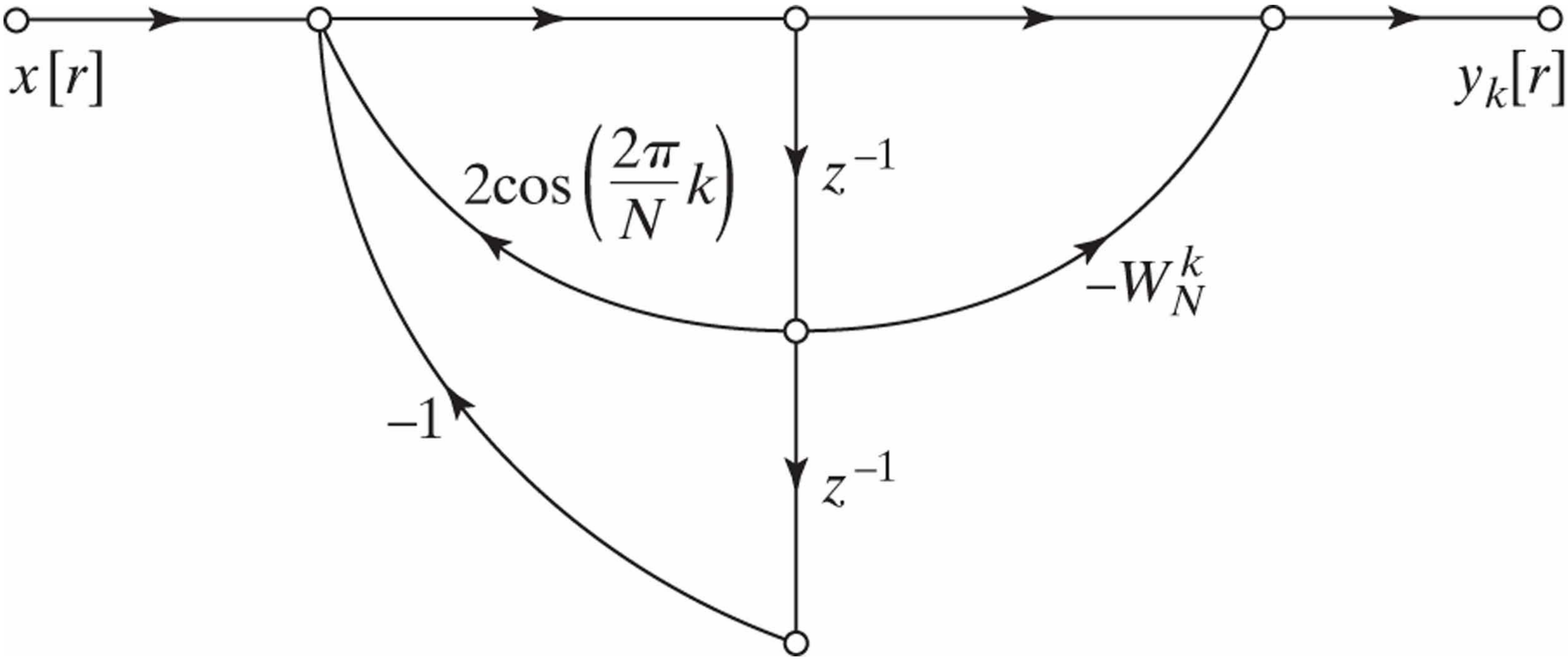
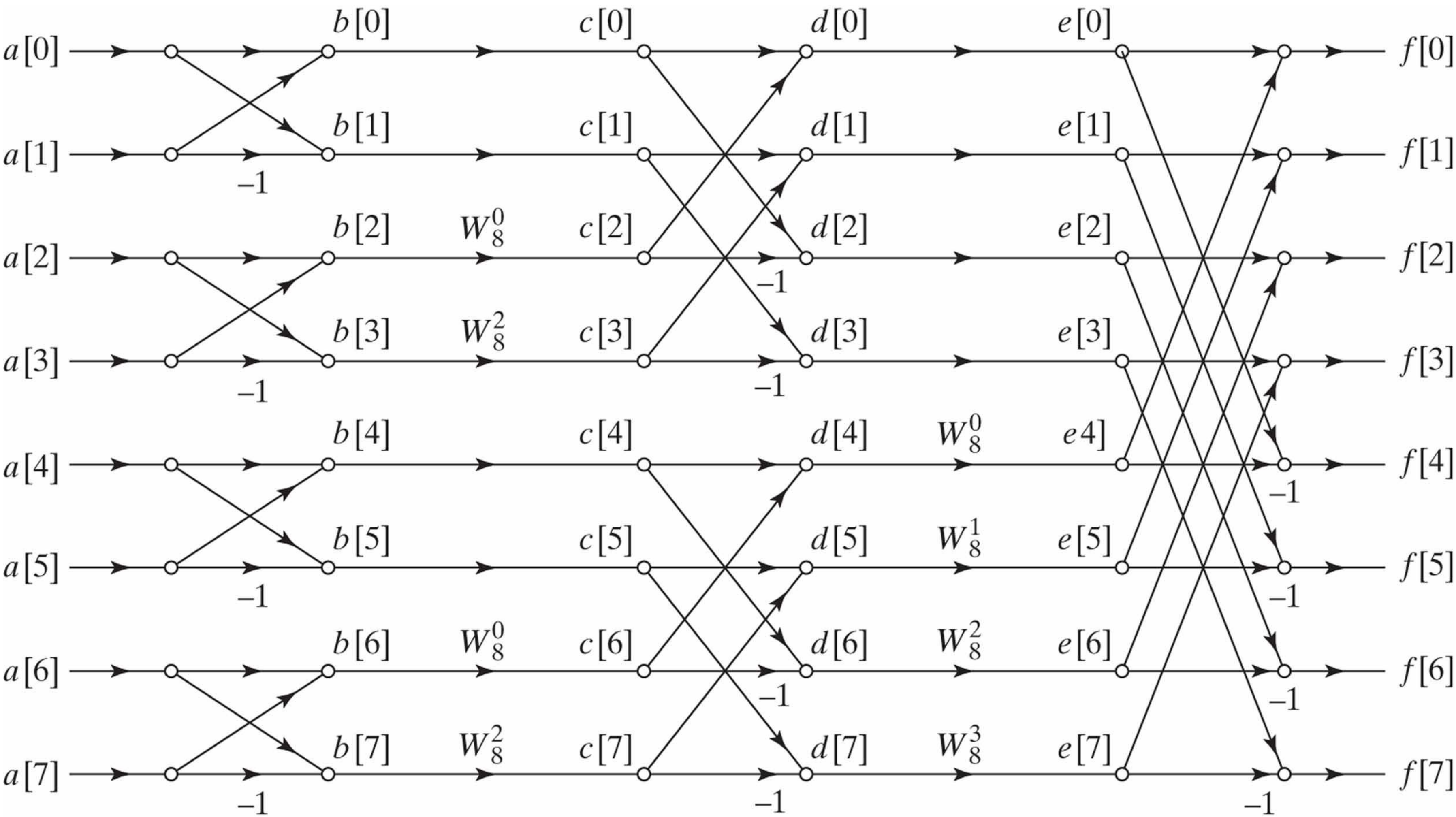


Figure P9.58



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W_8^3
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